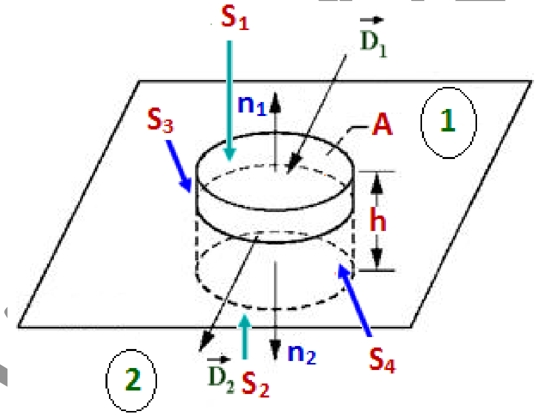
Boundary Conditions at Plane Interface between two Media:

In each of the two media, let us assume that the solution of Maxwell's equations that we desire is a plane wave, just as in an infinite media. At the boundary between the two-media, however, certain boundary conditions are to be met and these demand that there be definite relations between the boundary conditions which must be satisfied by field vectors $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{H}$ at the interface between media.

(i) Boundary Condition for Electric Displacement Vector $\vec{D}$:

We have from Maxwell's first equation $\vec{\nabla}.\vec{D} = \rho$. We now consider the interface of two dielectric media at which we also take a cylindrical pillbox-like surface $\mathbf{S}$ composed of S_1, S_2, S_3 and S_4 as shown in figure. Integrating over the pill-box-shaped volume V, we get $\int_V \operatorname{div}\vec{D}\, dv = \int_V \rho\, dv$



Again by Gauss's divergence theorem, we have

$$\oint_S \vec{D}\,.d\vec{s} = \int_V \rho\, dv$$

$$\text{i.e.} \int_{S_1} \vec{D}_1.\hat{n}_1 ds + \int_{S_2} \vec{D}_2.\hat{n}_2 ds + \int_{S_3} \vec{D'}_1.\hat{n'}_1 ds + \int_{S_4} \vec{D'}_2.\hat{n'}_2 ds = \int_V \rho\, dV$$

Where D_1 and D_2 are respective electric displacement vectors in medium – 1 and medium – 2 and n_1 and n_2 are respective unit normal at those two medium with respect to the end cross sections of the pill box as shown. If now $\vec{D}$ is bounded, letting the height of the pillbox h, approach zero, the third and fourth terms of the above equation vanishes and S_1 approaches S geometrically and the entire surface takes the form of A as shown in figure.

Hence in the limit $h \to 0$ we get $\lim_{h\to 0}\left[\int_{S_1} \vec{D}_1.\hat{n}_1 ds + \int_{S_2} \vec{D}_2.\hat{n}_2 ds\right] = \lim_{h\to 0}\int_V \rho\, dV$

If σ is the surface charge density at that interface then $\vec{D}_1.\hat{n}_1 S_1 + \vec{D}_2.\hat{n}_2 S_2 = \sigma A$

i.e. $(\vec{D}_1.\hat{n}_1 + \vec{D}_2.\hat{n}_2)A = \sigma A$ and then also for arbitrary surface area and $\hat{n}_1 = \hat{n}, \hat{n}_2 = -\hat{n}$ we get $\vec{D}_1.\hat{n} - \vec{D}_2.\hat{n} = \sigma$ i.e. $D_{1n} - D_{2n} = \sigma$

where D_{1n} and D_{2n} are the normal components of electric displacement vector in the two media. Thus we conclude that the normal component of electric displacement is not continuous at the interface but changes by an amount equal to the free surface charge density at the interface.

(ii) Boundary Conditions for Magnetic Induction $\vec{B}$:

We have from Maxwell's second equation is $\vec{B} = 0$. Again by taking volume integral over the entire volume of pillbox, we get $\int_V \text{ div } \vec{B} \, dv = 0$

Using Gauss's divergence theorem $\oint_S \vec{B} \, . d\vec{s} = 0$

$$\text{i.e.} \qquad \int_{S_1} \vec{B}_1 . \hat{n}_1 ds + \int_{S_2} \vec{B}_2 . \hat{n}_2 ds + \int_{S_3} \vec{B}'_1 . \hat{n}'_1 ds + \int_{S_4} \vec{B}'_2 . \hat{n}'_2 ds = 0$$

Considering $h \to 0$ as in earlier part, we note that third and fourth terms vanish while S_1 and S_2 approach each other; so that in the limit $h \to 0$, we get

$$\int_V (\vec{B}_1 . \hat{n}_1 + \vec{B}_2 . \hat{n}_2) \, ds = 0 (\text{Since } S_1 = S_2 = A)$$

Since surface is arbitrary, we get $\vec{B}_1 . \hat{n}_1 + \vec{B}_2 . \hat{n}_2 = 0$

For $\hat{n}_1$ and $\hat{n}_2$ in opposite sense $\hat{n}_1 = \hat{n}$ and $\hat{n}_2 = -\hat{n}$ and then we get

$\vec{B}_1 . \hat{n} - \vec{B}_2 . \hat{n} = 0$ $\quad$ i.e. $B_{1n} = B_{2n}$ I.e. the normal component of magnetic induction is continuous across the interface.

(iii) Boundary Condition for Electric Intensity Vector $\vec{E}$:

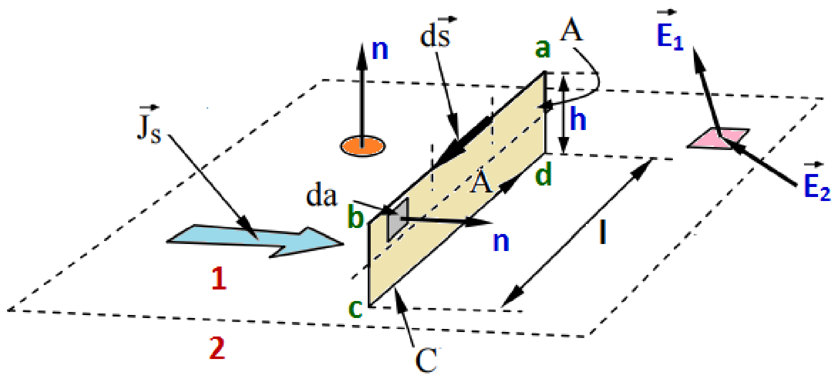
We again get from Maxwell's third equation is $\quad \text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

At the interface between two media, we now consider a rectangular loop $abcd$bounding a surface S as shown in figure. Integrating over the loop $abcd$, we get

$\int_S \text{curl } \vec{E} \, . \hat{n} \, ds = -\int_S \frac{\partial \vec{B}}{\partial t} \, . \hat{n} \, ds$. Now by using Stroke's theorem we get

$$\oint_{abcd} \vec{E} . d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} . \hat{n} \, ds \text{ or } \int_{ab} \vec{E}_1 . d\vec{l} + \int_{cd} \vec{E}_2 . d\vec{l} + \int_{bc \text{ and } da} \vec{E} . d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} . \hat{n} \, ds$$

If the loop is now shrunk by taking $h \to 0$ then along the interface the contribution to the integral from sides bc and da will vanish $\;$ i.e. $\quad \lim_{h \to 0} \int_{bc \text{ and } da} \vec{E} . d\vec{l} \to 0$



And also the surface integral on R.H.S. of above equation tends to zero provided that $\left(\frac{\partial B}{\partial t}\right)$ is finite everywhere. Thus in the limit $h \to 0$, we get

$$\left[\int_{ab} \vec{E}_1 . d\vec{l} + \int_{cd} \vec{E}_2 . d\vec{l}\right] = 0 \quad \text{Or} \quad E_{1t} l - E_{2t} l = 0 \quad \text{i.e.} \quad E_{2t} = E_{1t}$$

where E_{1t} and E_{2t} are the tangential components of the electric field in the two media. Here $E_{2t} = E_{1t}$ represents that tangential components of E must be continuous across the interface.

(iv) Boundary Condition for Magnetic Field Intensity $\vec{H}$:

Also we have from Maxwell's fourth equation is $\text{curl}\, \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Again by taking the surface integral over rectangular loop $abcd$ we get

$\int_S \text{curl}\, \vec{H} . \hat{n}\, ds = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) . \hat{n}\, ds$. Using Stoke's theorem we get

$\oint_{abcd} \vec{H} . d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) . \hat{n}\, ds$ or $\int_{ab} \vec{H}_1 . d\vec{l} + \int_{cd} \vec{H}_2 . d\vec{l} + \int_{bc\text{ and }da} \vec{H} . d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) . \hat{n}\, ds$

If the loop is taken compressed along the interface we get in the limit $h \to 0$, $\lim_{h\to 0} \int_{bc\text{ and }da} \vec{H} . d\vec{l} \to 0$ and $\lim_{h\to 0} \int \frac{\partial \vec{D}}{\partial t} . \hat{n}\, ds \to 0$

$$\text{i.e.} \quad \frac{\partial D}{\partial t} \text{ is bounded everywhere and } \lim_{h\to 0} \int_s \vec{J} . d\vec{s} \to J_{S\perp} l$$

Where $J_{S\perp}$ represents the components of surface current density perpendicular to the direction of H-component which is being matched.

The idea of surface current density is closely analogous to that of a surface charge density it represents a finite current in an infinitesimal layer. Then in the limit $h \to 0$,

$$\int_{ab} \vec{H}.d\vec{l} + \int_{cd} \vec{H}.d\vec{l} = J_{S\perp}l \quad \text{Or} \quad H_{1t}l - H_{2t}l = J_{S\perp}l \quad \text{i.e.} \quad H_{1t} - H_{2t} = J_{S\perp}l$$

Thus the tangential component of magnetic field intensity is not continuous at the interface; but changes by an amount equal to the component of the surface current density perpendicular to tangential component of $\mathbf{H}$.

If the surface current density is zero unless the conductivity is infinite; hence for finite conductivity $J_S = 0$; so $H_{1t} - H_{2t} = 0$ i.e. $H_{1t} = H_{2t}$

That is for one medium has infinite conductivity the tangential component of magnetic field intensity is continuous.