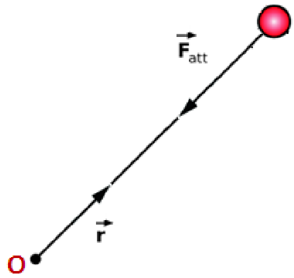
Characteristics of Central Force Field:

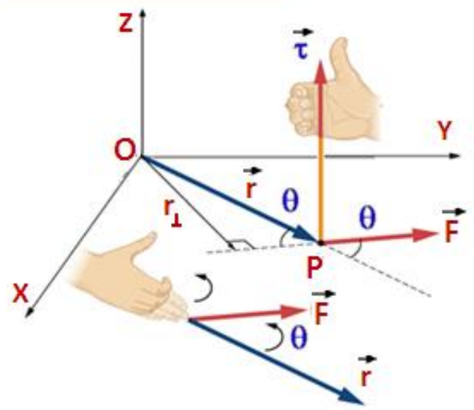


A central force is a particular force field which is always directed towards or away from a fixed point. That fixed point is called '**pole**' of the interaction and thus such central force may be **attractive** (i.e. negative) or **repulsive** (i.e. positive), which is always directed in the **radial sense.** The magnitude of this force solely depends on the distance $\mathbf{r}$ of the particle on which the force acts from the fixed point. Let us suppose that the force on a particle of mass $\mathbf{m}$ is $\vec{\mathbf{F}} = \mathbf{m}\,\vec{\mathbf{f}}(\mathbf{r})$. In that case we always have for central force $\vec{\mathbf{F}}$,

$$|\vec{\mathbf{F}}| = \mathbf{F} = \mathbf{mf(r)} = \mathbf{m}|\vec{\mathbf{f}}(\mathbf{r})| \quad \text{and also}$$

$$\vec{\mathbf{F}} = \mathbf{m}\,\vec{\mathbf{f}}(\mathbf{r}) = |\vec{\mathbf{F}}|(\pm\hat{\mathbf{r}}) = \pm\mathbf{mf(r)}.\hat{\mathbf{r}}$$

For Ordinary Force



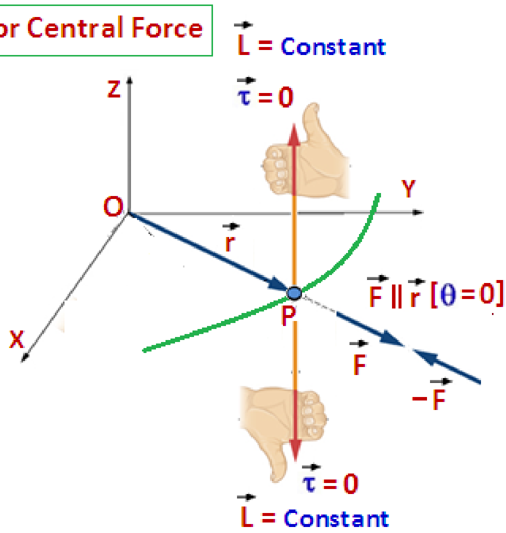
With this basic criteria of this central force we should immediately have

a) $\vec{\mathbf{f}}(\mathbf{r}) = \frac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2} = \frac{\vec{\mathbf{F}}}{\mathbf{m}} =$ Acceleration under central force directed towards or away from the fixed point.

b) Since $\vec{\mathbf{r}} \times \vec{\mathbf{F}} = \mathbf{0}$,

We have $\vec{\mathbf{r}} \times \vec{\mathbf{f}}(\mathbf{r}) = \mathbf{0}$ or $\vec{\mathbf{r}} \times \frac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2} = \mathbf{0}$

For Central Force



Again

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{r}} \times \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) = \left(\frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}} \times \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) + \left(\vec{\mathbf{r}} \times \frac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2}\right) = \mathbf{0} + \vec{\mathbf{r}} \times \frac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2}.$$

So we have $\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{r}} \times \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) = \mathbf{0}$

Integrating, we have, $\left(\vec{\mathbf{r}} \times \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) = \mathbf{h}$, where $\mathbf{h}$ is a constant. Again we have $\frac{1}{2}\left(\vec{\mathbf{r}} \times \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) = \frac{1}{2}\mathbf{h}$

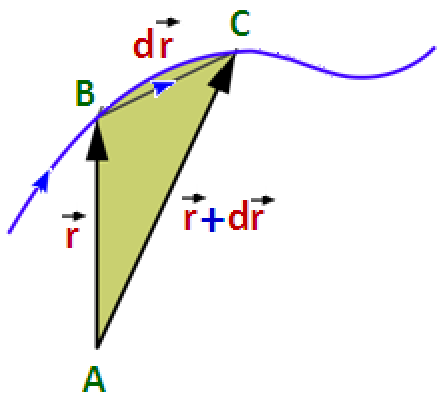
So we can conclude that for particle motion under central force, the area swept out by the radius vector in unit time for such particle motion will be constant. More clearly, the areal velocity is constant for motion of particle under a central force.

c) Since $\vec{\mathbf{r}} \times \vec{\mathbf{F}} = \mathbf{0}$ (i.e. the moment of this force about the fixed point is zero), the torque effective on that particle moving under central force will be zero.

That is $\quad \vec{\tau} = \frac{d\vec{L}}{dt} = 0 \implies \vec{L} = \text{Constant}$. This gives that the angular momentum of particle motion under central force will remain conserved. Again we have

$$\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = h \implies m\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = mh \implies \left(\vec{r} \times m\frac{d\vec{r}}{dt}\right) = mh \implies \vec{L} = \text{Constant}$$

d) It should also be noted that the vector product $\left(\vec{r} \times \frac{d\vec{r}}{dt}\right)$ is a vector normal to the plane determined by the fixed directions of $\vec{r}$ and $d\vec{r}$. The vector $m\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \left(\vec{r} \times m\frac{d\vec{r}}{dt}\right)$ is called the moment of momentum or the angular momentum about the fixed point. Thus we may conclude that for motion of a particle under a central force field, the angular momentum vector remains constant, and the orbit of the particle is such that for any value of $\vec{r}$ and $\frac{d\vec{r}}{dt}$ on it, there is a constant vector associated, and hence the orbit described will lie in a plane. Hence the motion of a particle under central force field must occur always in a plane.



e) For such central force $\vec{F} = |\vec{F}|(\pm\hat{r}) = \pm\frac{F}{r}\vec{r}$ we have $\vec{\nabla} \times \vec{F} = \pm\vec{\nabla} \times \left(\frac{F}{r}\vec{r}\right)$. Hence we get $\vec{\nabla} \times \vec{F} = \pm\left[\vec{\nabla}\left(\frac{F}{r}\right) \times \vec{r} + \left(\frac{F}{r}\right)\vec{\nabla} \times \vec{r}\right] = \pm\left[\vec{\nabla}\left(\frac{F}{r}\right) \times \vec{r} + 0\right] = \pm\left[\left\{\frac{1}{r}\vec{\nabla}(F) + F\vec{\nabla}\left(\frac{1}{r}\right)\right\} \times \vec{r}\right]$

Finally we get $\quad \vec{\nabla} \times \vec{F} = \pm\left[\left\{\frac{1}{r}.\frac{F'(r)}{r}\vec{r} - F\left(\frac{\vec{r}}{r^3}\right)\right\} \times \vec{r}\right] = 0$

Thus central force is always '**Irrotational**' and hence '**conservative**' and in that case we must have $\vec{F} = -\vec{\nabla}(U)$ and also usually $\quad F = -\frac{\partial U}{\partial r}$ where U is the potential energy.