Electron's Angular Momentum: Space Quantization:

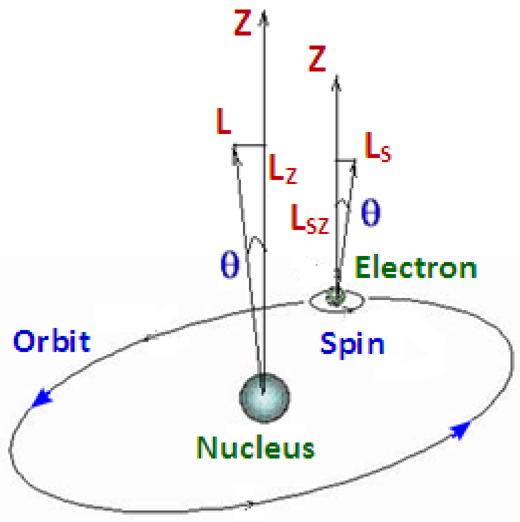
By this model if one expresses the angular momentum for orbital or spin motion of electron in vector form then the corresponding quantum numbers can also be expressed in vector form. As for example,

1. The angular momentum for orbital motion of electron is

$\vec{L} = \vec{l}\frac{h}{2\pi}$ **and** $|\vec{L}| = |\vec{l}|\frac{h}{2\pi} \Longrightarrow L = l\frac{h}{2\pi}$ **where** $L =$ **orbital angular momentum and** $l =$ **orbital angular momentum quantum number. Here we should note that for given principal quantum number** n**, the corresponding** l **−values are** $l = 0, 1, 2, 3, \ldots\ldots, (n-1)$

2. For spin motion of electron, its spin angular momentum is

$\vec{L}_S = \vec{s}\frac{h}{2\pi}$ **and** $|\vec{L}_S| = |\vec{s}|\frac{h}{2\pi} \Longrightarrow L_S = s\frac{h}{2\pi}$ **where** $L_S =$ **spin angular momentum and** $s =$ **spin angular momentum quantum number.**



Here we should mention that by Fermi – Dirac statistics, the value of spin angular momentum quantum number for electron will be only $s = \frac{1}{2}$ **(always)**

We also note that the scientist Ullenback and Goudsmith suggested the spin rotation of electron and with this consideration; they explained the observation of Stern Garlac experiment.

For this spin motion, electron has another magnetic moment which is called spin magnetic moment and it is given by $\vec{\mu}_s = g\mu_B\vec{s}$ **where** $g =$**gyro magnetic ratio,** $\mu_B =$ **Bohr Magnetron and from ESR spectroscopy (Electron spin resonance spectroscopy) it can be shown that** $g \approx 2$**.**

By this vector atom model, if external magnetic field (H) or magnetic induction (B) be applied to the atom then for angular momentum component $\vec{L}_z$ **along that applied magnetic field we can write that** $|\vec{L}_z| = m_l\frac{h}{2\pi}$ **where** m_l **is another quantum number called magnetic orbital angular momentum number.**

Again since $|\vec{L}_z| = LCos\theta$ **we should have** $m_l = l.Cos\theta$**. Also we have** $-1 \le Cos\theta \le +1$ **and thus for given orbital angular momentum quantum number** l **the number of possible discrete values of** m_l **will be** $(2l+1)$ **and these are given by**

$$\mathbf{m_l = l, (l-1), (l-2), \dots\dots 0. \dots, -l.}$$

On the other hand, for spin motion of electron, if $\vec{\mathbf{L}}_{sz}$ be the component of spin angular momentum along the applied magnetic field direction then we similarly have $|\vec{\mathbf{L}}_{sz}| = \mathbf{m_s}\frac{\mathbf{h}}{\mathbf{2\pi}}$ where $\mathbf{m_s}$ is magnetic spin quantum number.

Similar to the previous case of orbital motion, here we also have $\mathbf{m_s = s.Cos\theta}$.

Again since $\mathbf{-1 \leq Cos\theta \leq +1}$ so we must have $\mathbf{m_s = \pm\frac{1}{2}}$ where $\mathbf{m_s = +\frac{1}{2}}$ denotes up spin configuration and $\mathbf{m_s = -\frac{1}{2}}$ denotes down spin configuration. It is because for $\mathbf{m_s = +\frac{1}{2}}$ we have $\mathbf{s.Cos\theta = +\frac{1}{2} \Rightarrow Cos\theta = +ve}$ and $\boldsymbol{\theta} =$ acute angle ($\boldsymbol{\theta < 90}$).

Again $\mathbf{m_s = -\frac{1}{2}}$ we have $\mathbf{s.Cos\theta = -\frac{1}{2} \Rightarrow Cos\theta = -ve}$ and $\boldsymbol{\theta} =$ obtuse angle ($\boldsymbol{\theta > 90}$). This gives down spin. Now since for given $\mathbf{n}$ values, there exists $\mathbf{n}$ number of $\mathbf{l}$ values and for a particular $\mathbf{l}$ value, there are $\mathbf{(2l+1)}$ number of $\mathbf{m_l}$ values.

Again for given $\mathbf{m_l}$ value there exists $\mathbf{m_s = \pm\frac{1}{2}}$ for two spin orientation of the electron

Hence we can say that in nth energy state, there exist n numbers of $\mathbf{l}$ sub states. Again for each $\mathbf{l}$ sub state, there exists $\mathbf{(2l+1)}$ number of $\mathbf{m_l}$ sub sub states. Thus if N be the highest occupation of electron in nth state then by Pauli's exclusion principle we can write mathematically

$$\mathbf{N = \sum_{l=0}^{(n-1)} 2(2l+1) = 4\sum_{l=0}^{(n-1)} l + 2\sum_{l=0}^{(n-1)} (1) = 4\sum_{l=0}^{(n-1)} l + 2\sum_{l=1}^{n} (1)}$$
$$\mathbf{= 4(0+1+2+3+\cdots+(n-1)) + 2n = 4 \times \frac{(n-1)(n-1+1)}{2} + 2n}$$
$$\mathbf{= 2n(n-1) + 2n = 2n^2}$$

Here in this vector atom model we see that the orbital angular momentum of electron is given by $\vec{\mathbf{L}} = \vec{\mathbf{l}}\frac{\mathbf{h}}{\mathbf{2\pi}}$ and $|\vec{\mathbf{L}}| = |\vec{\mathbf{l}}|\frac{\mathbf{h}}{\mathbf{2\pi}} \Rightarrow \mathbf{L = l\frac{h}{2\pi}}$.

In classical concept, since the nth orbit is composed of n number of l – orbital's and the orbital angular momentum quantum number $\mathbf{l}$ has the possible values $\mathbf{l = 0, 1, 2, 3, \dots\dots, (n-1)}$, the orbital angular momentum of the electron cannot have any values but it should have a few specific values. This is called 'space quantization'.

Here we should mention that these specific values of electrons orbital angular momentum or angular momentum quantum numbers which is termed as space quantization, is strongly supported by quantum mechanical arguments.

By old quantum theory $L = l\frac{h}{2\pi}$, but it can be shown by new quantum mechanics or Schrödinger's quantum mechanics that $L = \sqrt{l(l+1)}\frac{h}{2\pi}$ with same $l-$ values, $l = 0, 1, 2, 3, \ldots\ldots, (n-1)$.

In quantum mechanics every classical parameter can be replaced by Hermitian operator obeying specific Eigen value equation (when that operator is the Eigen operator of that Eigen value equation and the Eigen value of that Eigen value equation will be that classical parameter). For energy operator, this Eigen value equation is actually Schrödinger's equation in new quantum mechanics (which will be discussed in quantum mechanics).

By solving Schrödinger's equation in hydrogen atom problem (which will be discussed in the part of quantum mechanics), it can be shown that the quantum state function is given by $\Psi = R_{nl}(r)Y_{lm}(\theta, \varphi)$ where $Y_{lm}(\theta, \varphi)$ is a part of that wave function, called **Spherical harmonics.**

It can also be shown that if $\widehat{L^2}$ be the square of angular momentum operator then $\widehat{L^2}Y_{lm}(\theta, \phi) = l(l+1)\left(\frac{h}{2\pi}\right)^2 Y_{lm}(\theta, \phi)$ where $l(l+1)\left(\frac{h}{2\pi}\right)^2$ is the eigenvalue of the operator $\widehat{L^2}$. So the Eigen value of angular momentum operator $\hat{L}$ should be

$$\sqrt{l(l+1)\left(\frac{h}{2\pi}\right)^2} = \sqrt{l(l+1)}\frac{h}{2\pi}.$$

Similarly if $\widehat{L_z}$ be the Z component of angular momentum operator then

$$\widehat{L_z}Y_{lm}(\theta, \phi) = m\frac{h}{2\pi}Y_{lm}(\theta, \phi).$$

Thus the angular momentum component operator has Eigen value $m\frac{h}{2\pi}$ where m is called magnetic orbital angular momentum quantum number in solving Schrödinger's equation for hydrogen atom problem, it is obtained that m has $(2l+1)$ possible values which are $m_l = l, (l-1), (l-2), \ldots\ldots 0. \ldots, -l$ as mentioned earlier.

With support of this, in vector atom model, the magnitude or the length of the orbital angular momentum vector is given by $|\vec{L}| = \sqrt{l(l+1)}\frac{h}{2\pi}$; $l = 0, 1, 2, \ldots, (n-1)$.

Thus if this angular momentum vector $\vec{L}$ makes angle θ with the specific Z direction then

$$\cos\theta = \frac{L_z}{|\vec{L}|} = \frac{m\left(\frac{h}{2\pi}\right)}{\sqrt{l(l+1)}\left(\frac{h}{2\pi}\right)} = \frac{m}{\sqrt{l(l+1)}}$$

The restriction on the direction of $\vec{L}$ to $(2l+1)$ components is known as Space quantization as already indicated.