Elasticity

1. Several Elastic Modules for Elastic Body:

Since for elastic behavior of the body, we have from Hooke's law

$$\text{Stress} \propto \text{Strain} \Rightarrow \text{Stress} = k_o.\text{Strain} \Rightarrow k_o \equiv \frac{\text{Stress}}{\text{Strain}} \text{ (Within elastic limit)}$$

We can have several elastic modules for several strain developed to the body. These elastic modules are now given below

a) Young's Modulus: This is defined by the ratio of tensile stress developed in the deformed body to the longitudinal strain within elastic limit. It is denoted by Y and thus mathematically it can be written as $Y \equiv \frac{\text{Tensile Stress Developed}}{\text{Longitudinal Strain}} = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$

b) Bulk Modulus: This is defined by the ratio of stress developed in the deformed body to the volume strain or bulk strain within elastic limit. It is denoted by B and thus mathematically it can be represented as $B \equiv \frac{\text{Stress Developed}}{\text{Bulk or Volume Strain}} = \frac{F/A}{\Delta V/V} = \frac{p}{\Delta V/V} = \frac{pV}{\Delta V}$

where p is excess pressure and other symbols have their usual meanings.

Again for initial pressure P of that elastic body if the excess pressure p be replaced by additional pressure ΔP, we then similarly have $B \equiv -\frac{\Delta P}{\Delta V/V} = -V\left(\frac{\Delta P}{\Delta V}\right)$ which is another representation of Bulk modulus. For this Bulk modulus we should note that

i) The reciprocal of this Bulk modulus is called Compressibility. It is mathematically given by

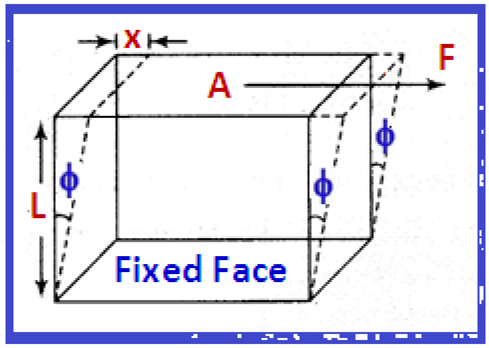
$$C \equiv \frac{1}{B} = -\frac{1}{V}\left(\frac{\Delta V}{\Delta P}\right)$$

ii) The isothermal Bulk modulus is equal to the initial pressure of that system. So in that case we have $B_{isothermal} = P =$ Initial pressure of the system

iii) The adiabatic Bulk modulus is equal to the initial pressure multiplied by $\gamma \left(\equiv \frac{C_P}{C_V}\right)$ of that system. So in that case we have $B_{adiabatic} = \gamma P = \gamma \times$ Initial pressure.

c) Shearing Angle or Angle of Shear:

When a tangential force is applied to a body then the angle of rotation of the transverse surface of that body with respect to the applied force is called angle of shear or shearing angle. It is denoted by φ and it is in general small. The significance of this shearing angle is that it will give the direct estimation of the shape strain of the body under application of the external tangential force. So we should have



$$\text{Shape strain} \equiv \varphi = \tan\varphi$$

This shearing angle is a dimensionless parameter which is expressed in practical unit of radian. It can also be shown that for the shape strain of the body under tangential force, this shearing angle is twice the longitudinal strain of the diagonal of a square type body.

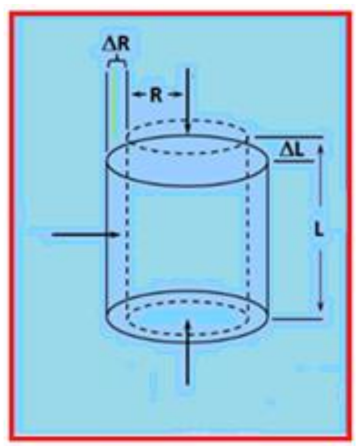
d) Modulus of Rigidity:

This is defined by the ratio of stress developed in the deformed body to the shape strain within elastic limit. It is denoted by η and thus mathematically it can be represented as

$$\eta \equiv \frac{\text{Stress Developed}}{\text{Shape Strain}} = \frac{\text{Stress Developed}}{\text{Shearing Angle}} = \frac{F/A}{\varphi} = \frac{F}{A\varphi}$$

e) Axial Modulus:

This is defined by the ratio of tensile stress developed in the deformed body to the longitudinal strain without any lateral strain within elastic limit. It is denoted by χ and thus mathematically it can be represented as

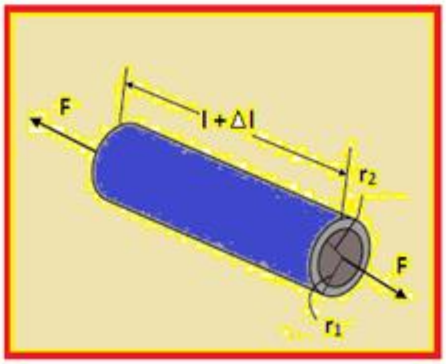


$$\chi \equiv \frac{\text{Tensile Stress Developed}}{\text{Longitudinal Strain without any Lateral Strain}} = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$$

2. Poisson's Ratio:

For an elastic body, if linear or longitudinal strain be made to a body, then along with that longitudinal strain of that body, lateral strain will also appear in the transverse sense. The ratio of that lateral strain to the longitudinal strain within elastic limit is called Poisson's ratio. It is an elastic constant which is denoted by σ.

For mathematical representation of this ratio, let us now consider a cylindrical body of radius $\mathbf{r}$ and length $\mathbf{l}$. If its length is made increased to $(\mathbf{l}+\Delta\mathbf{l})$ then its radius will reduce to $(\mathbf{r}-\Delta\mathbf{r})$. In that case, we have



$$\text{lateral strain} \equiv \frac{(\mathbf{r}-\Delta\mathbf{r})-\mathbf{r}}{\mathbf{r}} = -\frac{\Delta\mathbf{r}}{\mathbf{r}}$$

And $\quad \text{longitudinal strain} \equiv \frac{(\mathbf{l}+\Delta\mathbf{l})-\mathbf{l}}{\mathbf{l}} = \frac{\Delta\mathbf{l}}{\mathbf{l}}$

Hence Poison's Ratio of that elastic body is given by

$$\sigma \equiv \frac{\text{lateral strain}}{\text{longitudinal strain}} = -\frac{\Delta r/r}{\Delta l/l}$$

The basic characteristics of this ratio are

i) It is an elastic constant but not an elastic modulus.

Ii) It has no unit or dimension

iii) It can never be negative

iv) For volume of the body constant, the maximum value of it is 0.5

v) For an elastic body without any shape strain, theoretically we have $\sigma \to \infty$

vi) It has mathematical limit of magnitude $-1 \leq \sigma \leq 0.5$ but in reality we have $0 < \sigma \leq 0.5$

vii) It can never be zero in reality

3. Relation among Several Elastic Modules:

The relation among several elastic modules can mathematically be derived. These relations are now given by

$Y = 3B(1-2\sigma) \to \; : \; Y, B, \sigma$ relation $\quad Y = 2\eta(1+\sigma) \to : Y, \eta, \sigma$ relation, $\frac{3B}{2\eta} = \frac{(1+\sigma)}{(1-2\sigma)} \to \; : \; B, \eta, \sigma$ relation $\quad \chi = \frac{Y(1-\sigma)}{(1+\sigma)(1-2\sigma)} \to \; : \; \chi, Y, \sigma$ relation and $\chi = B + \frac{4}{3}\eta \to : \chi, B, \eta$ relation.

4. Relation between Volume Strain and Longitudinal Strain of an Elastic Body:

Here we consider a cylindrical body having radius $\mathbf{r}$ and length $\mathbf{l}$. So the volume of that cylindrical body will be $V = \pi r^2 l$ and in that case the volume change is given by $\Delta V = \pi(r^2\Delta l + l.2r\Delta r)$

Thus for such elastic body, the volume strain will be $\frac{\Delta V}{V} = \frac{\pi(r^2\Delta l + l.2r\Delta r)}{\pi r^2 l} \Longrightarrow \frac{\Delta V}{V} = \left(\frac{\Delta l}{l} + 2\frac{\Delta r}{r}\right)$

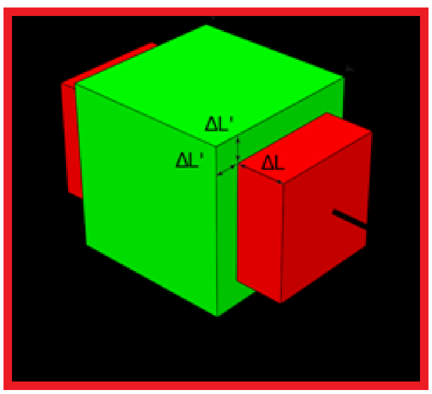
Again we have Poisson's ratio is given by $\sigma = -\frac{\Delta r/r}{\Delta l/l} \Rightarrow \frac{\Delta r}{r} = -\sigma\frac{\Delta l}{l}$.

Thus we can have $\frac{\Delta V}{V} = \frac{\Delta l}{l}(1-2\sigma)$. This is the general relation between volume strain and longitudinal strain of an elastic body. From this relation we also see that if the volume of that elastic body remain constant i.e. volume strain of that body becomes zero than we should have $\Delta V = 0 \Rightarrow \frac{\Delta V}{V} = \frac{\Delta l}{l}(1-2\sigma) = 0$ and $1-2\sigma = 0$ and finally $\sigma = \frac{1}{2}$.

Hence the conclusion is that if the volume of an elastic body remains constant then Poisson's ratio will be $\frac{1}{2}$.

5. Theoretical Negative Value of Poisson's Ratio:

For simple consideration, if we take that for elasticity, the shape of the body remains unchanged i.e. no shape change occurs then the corresponding shearing angle will be zero and the modulus of rigidity of that elastic medium will be infinite. Because since modulus of rigidity is given by $\eta = \frac{F}{A\theta}$ then for $\theta = 0$ we have $\eta \rightarrow \infty$. Again for elasticity, we have one of the most useful relation among elastic module, $Y = 2\eta(1+\sigma)$ we have for $\theta = 0, \eta = \frac{Y}{2(1+\sigma)} \rightarrow \infty$ and then $1+\sigma = 0 \Rightarrow \sigma = -1$.



Thus we see that mathematically if the shape of the body remains unchanged then Poisson's ratio will be -1.

6. Derivation of Mathematical Limit of Magnitude of Poisson's Ratio:

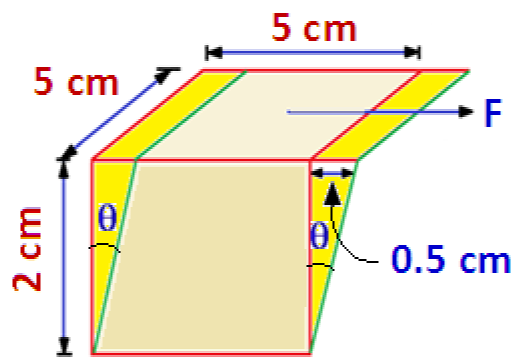
For elasticity, we have two most useful relations among elastic module, which are $Y = 2\eta(1+\sigma)$ and $Y = 3B(1-2\sigma)$, and in that case we can write down $B(1-2\sigma) = 2\eta(1+\sigma) \Rightarrow \frac{3B}{2\eta} = \frac{1+\sigma}{1-2\sigma}$.

Again we have $\frac{3B}{2\eta} = \frac{1+\sigma}{1-2\sigma} = \frac{1-(-\sigma)}{1-2\sigma} = \text{positive}$ because elastic module can never be negative. Thus from this relation we should have $-\sigma < 1$ and < 1. That is $-1 < \sigma$ and $\sigma < \frac{1}{2}$. Again we know that for volume of the elastic body constant, $\sigma = \frac{1}{2}$ and for no shape change, $\sigma = -1$. So in general we should have $-1 \leq \sigma \leq \frac{1}{2}$. This is the mathematical limit of magnitude of Poisson's ratio.

Solved Problems

1. A solid block is $5 \text{ cm} \times 5 \text{ cm} \times 2 \text{ cm}$ when unstressed. A force of 0.25 N acts tangentially on the upper face and displaces it by 0.5 cm relative to the fixed lower surface as shown in the figure. Find the (i) shearing strain, (ii) shearing stress and (iii) shear modulus.

Ans: (i) Shearing strain, $\theta = \frac{\Delta L}{L} = \frac{0.5}{2} = 0.25$



(ii) Shearing stress $= \frac{F}{A} = \frac{0.25}{25 \times 10^{-4}} = 100 \text{ Nm}^{-2}$

(iii) Shear modulus

$$G = \frac{\text{Shearingstess}}{\text{Shearingstrain}} = \frac{100}{0.25} = 400 \text{ Nm}^{-2}$$

2. A composite wire of uniform diameter 3.0 mm consists of a copper wire of length 2.2 m and steel wire of length 1.6 m stretches under a load by 0.7 mm. Calculate the load, given that young's modulus for copper is 1.1×10^{11}Pa and that for steel is 2.0×10^{11}Pa. $(1 \text{ Pa} = 1 \text{ Nm}^{-2})$

Ans: For copper wire $L_C = 2.2 \text{ m}, Y_C = 1.1 \times 10^{11} \text{Nm}^{-2}$;

For steel wire, $L_S = 1.6 \text{ m}, Y_S = 2.0 \times 10^{11} \text{Nm}^{-2}$ and $\Delta L_C + \Delta L_S = 0.7 \text{ mm} = 7 \times 10^{-4} \text{m}$
$r = \frac{3}{2} \times 10^{-3} = 1.5 \times 10^{-3} \text{m}$

From $Y = \frac{\text{Stress}}{\text{Strain}}$ we get $\text{Stress} = Y \times \text{Strain} = Y \times \frac{\Delta L}{L}$.

The stress is equal for the composite wire

Hence $Y_C \times \frac{\Delta L_C}{L_C} = Y_S \frac{\Delta L_S}{L_S}$ Or $\frac{\Delta L_C}{\Delta L_S} = \frac{Y_S}{Y_C} \times \frac{L_C}{L_S} = \frac{2 \times 10^{11}}{1.1 \times 10^{11}} \times \frac{2.2}{1.6} = 2.5$ and $\Delta L_C = 2.5 \, \Delta L_S$

But $\Delta L_C + \Delta L_S = 7 \times 10^{-4}$ Or $2.5 \, \Delta L_S + \Delta L_S = 7 \times 10^{-4}$

Thus $\Delta L_S = 2 \times 10^{-4} \text{m}$ and $\Delta L_C = 5 \times 10^{-4} \text{m}$.

From $Y_C = \frac{F}{\pi r^2} \cdot \frac{L_C}{\Delta L_C}$ So we get $F = Y_C \times \pi r^2 \times \frac{\Delta L_C}{L_C}$

$$F = 1.1 \times 10^{11} \times 3.14 \times (1.5 \times 10^{-3})^2 \times \frac{5 \times 10^{-4}}{2.2} = 176.8\ \mathrm{N}$$

3. A copper wire of length 5.0 m and cross sectional area $2.0\ \mathrm{mm^2}$ elongates by 2.5 mm when subjected to a stretching force. Find the density of elastic potential energy stored in stretched wire. Young's modulus of copper is $1.20 \times 10^{11} \mathrm{Nm^{-2}}$.

Ans: Given: $L = 5.0\ \mathrm{m}, A = 2 \times 10^{-6} \mathrm{m^2}, \Delta L = 2.5\ \mathrm{mm}$.

$$\mathrm{Strain} = \frac{\Delta L}{L} = \frac{2.5 \times 10^{-3}}{5} = 0.5 \times 10^{-3}$$

And Stress $= Y \times \mathrm{Strain} = 1.20 \times 10^{11} \times 0.5 \times 10^{-3} = 6 \times 10^7 \mathrm{Nm^{-2}}$

Density of elastic energy stored in the wire

$$= \frac{1}{2} \times \mathrm{Stress} \times \mathrm{Strain} = \frac{1}{2} \times 6 \times 10^7 \times 0.5 \times 10^{-3} = 1.5 \times 10^4 \mathrm{Jm^{-3}}$$

4. What is the density of water at a depth where the pressure is 80.0 atm? Given that its density at the surface is $0.03 \times 10^3 \mathrm{kgm^{-3}}$? Compressibility of water is $45.8 \times 10^{-11} \mathrm{Pa^{-1}}; 1\ \mathrm{Pa} = 1\ \mathrm{Nm^{-2}}$

Ans: Given: Compressibility$= \frac{1}{B} = 45.8 \times 10^{-11} \mathrm{Pa^{-1}}$.

Increase in pressure $\Delta P = 80 - 1 = 79\ \mathrm{atm.} = 79 \times 1.03 \times 10^5 \mathrm{Nm^{-2}}$.

From$B = \Delta P \,.\frac{V}{\Delta V}$ and $\frac{\Delta V}{V} = \frac{\Delta P}{B} = 79 \times 1.03 \times 10^5 \times 45.8 \times 10^{-11} = 3.665 \times 10^{-3}$

But$\frac{\Delta V}{V} = \frac{V - V'}{V} = \frac{M/\rho - M/\rho'}{M/\rho} = 1 - \frac{\rho}{\rho'}$, Thus $\rho' = \frac{\rho}{1 - \Delta V/V} = \frac{1.03 \times 10^3}{1 - 3.665 \times 10^3} = 1.034 \times 10^{-3} \mathrm{kgm^{-3}}$

5. Two wires of the same materials have their lengths in ratio of 2 : 3 and radii in the ratio of 2 : 1. Find the ratio of the stretching forces acting on them when (i) Strains produced in both are equal (ii) elongations produced in both are equal (iii) stress produced in both are equal.

Ans: Given: $\frac{L_1}{L_2} = \frac{2}{3}, \frac{r_1}{r_2} = \frac{2}{1}$ and $Y_1 = Y_2$

From $Y = \frac{F}{\pi r^2} \,.\frac{L}{\Delta L}$ (i) $\frac{F_1}{\pi r_1^2} \,.\frac{L_1}{\Delta L_1} = \frac{F_2}{\pi r_2^2} \,.\frac{L_2}{\Delta L_2}$, But $\frac{L_1}{\Delta L_1} = \frac{L_2}{\Delta L_2}$ (given) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2} = \frac{4}{1}$

(ii) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2} \,.\frac{L_2}{L_1} = \frac{4}{1} \times \frac{3}{2} = \frac{6}{1}$. (As $\Delta L_1 = \Delta L_2$) (iii) $\frac{F_1}{\pi r_1^2} = \frac{F_2}{\pi r_2^2}$ and then $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2} = \frac{4}{1}$

6. Calculate the value of Poisson's ratio of the material of a wire whose volume remains constant under an external normal stress.

Ans: Let L and R be the initial length and radius of the wire.

Volume of the wire $V = \pi R^2 L = \text{constant}$.

Differentiating both side $0 = \pi(2R\Delta R \times L + R^2 \Delta L)$. Dividing by $R^2 L$ we have $\frac{2\Delta R}{R} + \frac{\Delta L}{L} = 0$

$\frac{2\Delta R}{R} = -\frac{\Delta L}{L}$ and $\frac{\Delta R/R}{\Delta L/L} = \frac{1}{2}$(Negative sing only indicates that as the length increases, the radius decreases) Hence Poisson's ratio $\sigma = 0.5$.

7. A cube of copper of edge 10 cm is subjected to a hydraulic stress of $10^8 Nm^{-2}$. Calculate the (i) change in volume of the cube and (ii) volume strain. Given: Bulk modulus of copper $(B) = 14 \times 10^{10} Nm^{-2}$

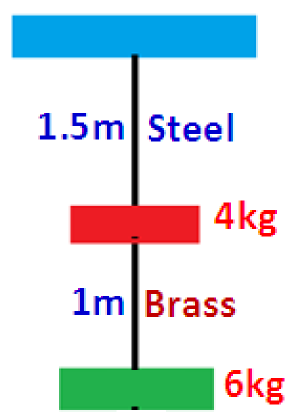
Ans: Given:$l = 10$ cm, $P = 10^8 Nm^{-2}$(i) From Bulk modulus, $= \frac{\text{Normalstress (P)}}{\text{Volumestrain}}$

$$\text{Volume strain } \frac{P}{B} = \frac{10^8}{14 \times 10^{10}} = 7.1 \times 10^{-4}$$

(ii) Volume strain $= \frac{\Delta V}{V}$ where $V = 10^3 cm^3 = 10^{-3} m^3$ Thus $7.1 \times 10^{-4} = \frac{\Delta V}{10^{-3}}$

$$\Delta V = 7.1 \times 10^{-7} m^3 = 0.71 \; cm^3$$

8. Two wires of diameter 0.25 cm each, one made of steel and other made of Brass are loaded as shown in the figure. The loaded length of steel wire is 1.5 m and that of brass wire is 1.0 m. compute the elongations of steel and brass wires. Young's modulus of steel 2.0×10^{11}Pa, Young's modulus of brass 0.91×10^{11}Pa



Ans: From $Y = \frac{MgL}{\pi r^2 \Delta L}$, $\Delta L = \frac{MgL}{\pi r^2 Y}$;

For steel wire Total load $= 6 + 4 = 10$ kgwt

$$\Delta L = \frac{10 \times 9.8 \times 1.5}{3.14 \times (0.125 \times 10^{-2})^2 \times 2 \times 10^{11}} = 1.5 \times 10^{-4} m$$

For brass wire $\Delta L = \frac{6 \times 9.8 \times 1}{3.14 \times (0.125 \times 10^{-2})^2 \times 0.91 \times 10^{11}} = 1.3 \times 10^{-4} m$

9. A 10 m long rubber string is suspended from a rigid support at its one end. Calculate the extension in the string due to its own weight. The density of rubber $1.5 \times 10^3 kgm^{-3}$.Take $g = 10\ ms^{-2}$. Young's modulus of rubber $= 5 \times 10^6 Nm^{-2}$.

Ans: Tensile stress $= \frac{F}{A} = \frac{Mg}{A} = \frac{LA\rho g}{A} = L\rho g = 10 \times 1.5 \times 10^3 \times 10 = 1.5 \times 10^5 Nm^{-2}$.

Longitudinal strain $= \frac{\Delta L}{L}$

From $Y = \frac{Stress}{Strain} = 1.5 \times 10^5 \times \frac{L}{\Delta L}$ we get $\Delta L = \frac{1.5\times10^5\times10}{5\times10^6} = 0.30\ m$

10. Compute the bulk modulus of water from the following data: Initial volume = 100.0 litre, Pressure increase = 100.0 atm, Final volume = 100.5 liter. Compare the bulk modulus of water with that of air. Explain in simple terms why the ratio is so large.

Ans: Given: $\Delta P = 100 \times 1.013 \times 10^5 Nm^{-2}; \Delta V = 100.5 - 100.0 = 0.5\ litre$

Bulk modulus of water $B_w = \Delta P \frac{V}{\Delta V} = \frac{101.3\times10^5\times100\times10^{-3}}{0.5\times10^{-3}} = 2.026 \times 10^9 Nm^{-2}$

The bulk modulus of air at constant temperature $B_a = P = 1.013 \times 10^5 Nm^{-2}$

$$[As\ \ PV = Constant,\ \ since\ \ P\Delta V + V\Delta P = 0, \Delta P\,.(V/\Delta V) = P]$$

Thus $\frac{B_w}{B_a} = \frac{2.026\times10^9}{1.013\times10^5} = 2 \times 10^4$. The ratio is as large as water is almost incompressible but air is highly compressible.

11. A wire of length 2.5 m and cross sectional area $1.5\ mm^2$ gets elongated by 1 mm when loaded weight W. Calculate the elongation produced in another wire of the same metal but of length 3 m and area of cross section $3\ mm^2$ when loaded by the weight 5 W.

Ans: From $Y = \frac{FL}{A\Delta L}$, $\Delta L = \frac{FL}{AY}$, $\therefore \frac{\Delta L_2}{\Delta L_1} = \frac{F_2 L_2}{A_2 Y} \times \frac{A_1 Y}{F_1 L_1} = \frac{F_2}{F_1} \times \frac{L_2}{L_1} \times \frac{A_1}{A_2}$ Thus $\frac{5W}{W} \times \frac{3}{2.5} \times \frac{1.5}{3} = 3$

$$So\ we\ get\ \Delta L_2 = 3 \times \Delta L_1 = 3 \times 1 = 3\ mm$$