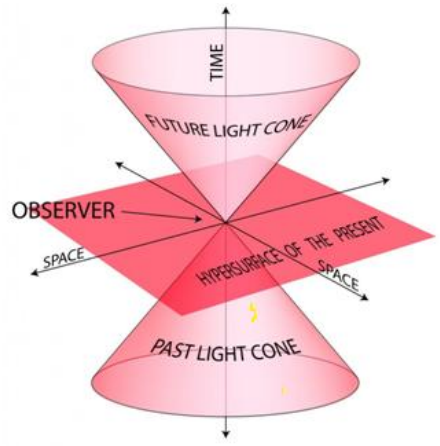
The Invariant Interval: Light Cone and World Lines: Space-Time Diagrams:

As we have already discussed before about the 4 dimensional Minkowski space which is basically replaced by a two dimensional Argand (Complex) plane with one space axis (x) and another time axis (ct) where the other two coordinates are made suppressed, here we should also note a few points about this original 4 – dimensional Minkowski space which are

a) In this 4 dimensional space, any event is present by a point, called 'event point' or 'world point'. This point must have 4 coordinates among of which three coordinates are space coordinates and one is time coordinate.



b) The position of any world point in this space – time region is called 4 – position and it is denoted by r_μ. To make $|r_\mu|^2$ Lorentz invariant, the time axis of this 4 – dimensional space is taken as imaginary and is expressed as 'ict' where $i = \sqrt{-1}$.

Thus we should note that the four position of a world point is presented as $r_\mu = (x, y, z, ict)$

c) In Minkowski space, the four position of event point or world point about the origin is expressed as $r_\mu = (x_1, x_2, x_3, x_4)$ where usually $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$ and thus we have $r_\mu^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x^2 + y^2 + z^2 - c^2t^2$ where for two Lorentz frames we must have $|r_\mu|^2 = |r'_\mu|^2$.

d) In equivalent two dimensional Argand plane for original 4 – dimensional Minkowski space, the space – time interval between two event points is given by

$$\Delta S = \sqrt{(\Delta x)^2 - c^2(\Delta t)^2} \text{ i.e. } (\Delta S)^2 = (\Delta x)^2 - c^2(\Delta t)^2$$

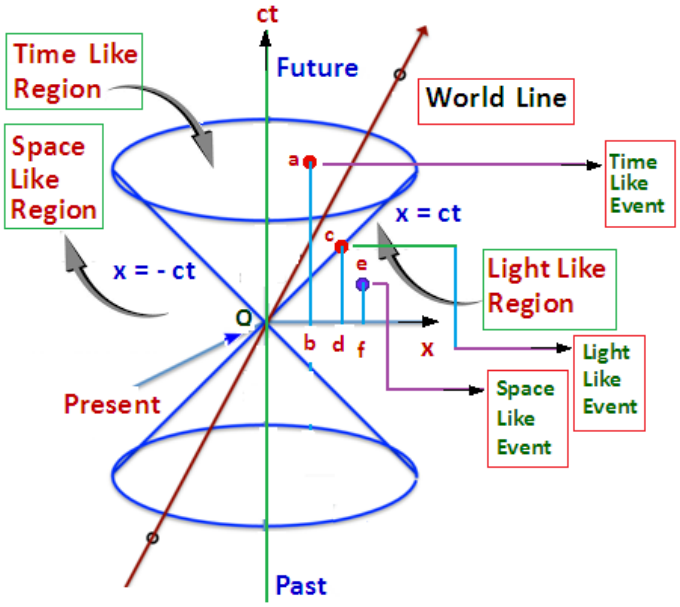
But for any signal travelling with velocity 'v' in three space in time Δt along x – axis (space axis), we have $\Delta x = v\Delta t < c\Delta t$ for $v < c$. In that case ΔS will become imaginary which is meaningless. For this reason, the space – time interval between two event points or world points as taken in equivalent two dimensional Argand plane is defined as

$\Delta S = \sqrt{(c\Delta t)^2 - (\Delta x)^2}$ I.e. $(\Delta S)^2 = (c\Delta t)^2 - (\Delta x)^2$ and this ΔS is called 'invariant interval' which must be Lorentz invariant.

e) Any curve connecting two world points in this 4 – dimensional space – time region is called 'world line'

On the basis of a few characteristics of this 4 – dimensional space – time region as mentioned above, we should note that if we now draw two world lines, each with 45° slope angle as presented by two equations $x = ct$ and $x = -ct$ then we will get two right

circular cones with their vertices in contact at the origin for complete rotation of these two world lines as shown in space – time diagram.



These two cones as obtained are called 'light cones' and the upper light cone associated with positive time axis is called 'Future Light Cone' and the light cone associated with negative time axis is called 'Past Light Cone'. The specialities of these two light cones are a) each light cone is bounded by the world lines $x = \pm ct$ b) for any two events taken on the surface of upper light cone (since in reality, event at negative time i.e. past event cannot be connected or reached) or world line, the invariant interval $\Delta S = 0$ for $x = \pm ct \Rightarrow \Delta x = \pm c\Delta t \Rightarrow |\Delta x| = c\Delta t$ and in that case we immediately have $\frac{|\Delta x|}{\Delta t} = v = c$. Thus any two events on the surface of upper light cone or on the world line $x = \pm ct$ in upper region can only be connected by a light signal. That is why any event taken on the surface of upper future cone is called 'Light Like Event' and the surface region on the upper future cone is called 'Light Like Region'. As shown in space – time diagram, the event point 'c' is a 'light like event'

Here we should note that the origin of this space time diagram at $t = 0$ is called 'present' or 'present moment' which basically connect 'past' with 'future'. It is obvious that any event in past can be connected with another event in future with a world line which must pass through the origin i.e. the 'present moment'.

Suppose we now take an event point 'a' within the upper future cone. For this event point or world point we see that $c\Delta t > \Delta x \implies c > \frac{\Delta x}{\Delta t} \Rightarrow c > v$. So we can connect any event in future with the present moment i.e. with the origin by a signal travelling with velocity less than that of light. Such event within future light cone is called 'Time Like Event' and the region within that future light cone is called 'Time Like Region'.

On the other hand, if we take another event point 'e' outside the upper future cone but in the upper region of space axis in association with the positive time axis then for this event point or world point we see that $c\Delta t < \Delta x \implies c < \frac{\Delta x}{\Delta t} \Rightarrow c < v$ which is 'absurd' in reality.

So we can never connect any event in future outside future light cone with the present moment i.e. with the origin since no signal can travel with velocity greater than that of

light in free space. Such event outside future light cone is called 'Space Like Event' and that forbidden region outside future light cone is called 'Space Like Region'.