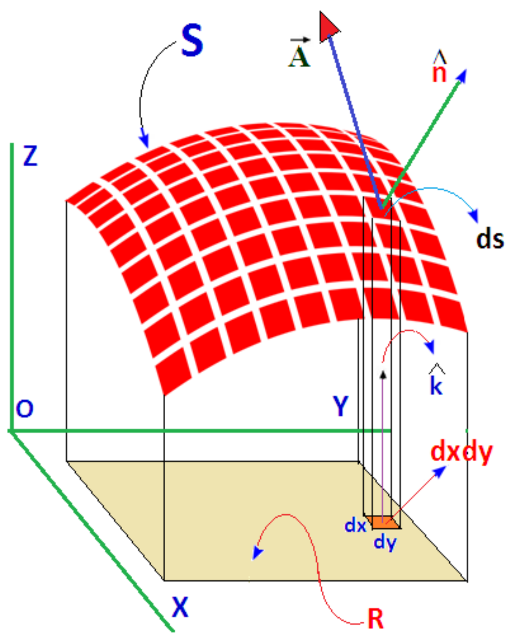
Concept of Surface Integration of a Field Vector:

We now consider a surface $\mathbf{S}$ which can now be divided into a large number of small elementary surface segments as shown in figure. Here we take such a surface element ds where is the unit normal to it. $\vec{A}$ is the field vector at the location of that surface element and since any field vector indicates the flow of filed lines or tube of flow, the quantity $\vec{A}.d\vec{s}$ (where $\vec{A}.\hat{n}$ gives the component of the field vector $\vec{A}$ along the sense of surface element ds) is a measure of flux out flow i.e. the out flow of the physical quantity in the sense normal to the surface area.

Thus the net flow of outward flux through the whole surface $\mathbf{S}$ is given by $\int_S \vec{A}.d\vec{s} = \int_S \vec{A}.\hat{n}ds$ and it is the surface integral of the field vector over the surface $\mathbf{S}$. Basically for the equation $\Phi(x, y, z) = \text{constant}$ of the surface $\mathbf{S}$, the unit normal at any point on this surface will be $\hat{n} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}$.



But the problem is that for surface area $\mathbf{S}$ of any arbitrary shape, it is in many cases, difficult to recognize the limit of integration for evaluation of the surface integral $\int_S \vec{A}.\hat{n}ds$ for any given field vector. Then in that case usually for evaluation of surface integral of field vector over any surface, the integration is made carried out over the projection surface taken on any of xy or yz or xz – plane .

As shown in figure, consider that for the given surface $\mathbf{S}$, its projection (shadow region) R is taken in xy – plane. Thus obviously, the projection surface of the elementary surface ds on that xy – plane must be "dxdy" as shown. Since this projection surface element "dxdy" has unit normal $\hat{k}$ (as it is in xy – plane), it is basically the component of ds on xy – plane.

So we can now write $dxdy = dsCos\theta \implies ds = \frac{dxdy}{Cos\theta} = \frac{dxdy}{|\hat{n}.\hat{k}|}$.

Hence finally the surface integral of that field vector $\vec{A}$ over the surface $\mathbf{S}$ is given by

$$I = \int_S \vec{A}.\hat{n}ds = \iint_{xy} \vec{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$$

This is the surface integral of the field vector $\vec{A}$ over the surface $\mathbf{S}$.

As for example, let us solve a problem 5.58 (a) (Supplementary Problem 5.58 (a), Spiegel) where the field vector is given by $\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and the surface S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$

Thus in the case we have for $\Phi = 2x + y$, $\hat{n} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|} = \frac{2\hat{i}+\hat{j}}{\sqrt{5}}$ and by taking projection on xz – plane, we get the surface integral

$$I = \int_S \vec{A}.\hat{n}ds = \iint_{xz} \vec{A}.\hat{n}\frac{dxdz}{|\hat{n}.\hat{j}|} = \iint_{xz} \left(\frac{2y+2x}{\sqrt{5}}\right)\frac{dxdz}{\frac{1}{\sqrt{5}}} = 2\iint_{xz} (y+x)\frac{dxdz}{\frac{1}{\sqrt{5}}} = 2\iint_{xz} (6-x)dxdz$$

Finally we get $I = \int_S \vec{A}.\hat{n}ds = 12\int_0^4 dz \int_0^3 dx - 2\int_0^4 dz \int_0^3 xdx = 144 - 36 = 108.$