Properties of Laplace's Transform:

a) Change of Scale Property:

By this property, if $L[f(x)] = L(s)$, then $L\,[f(ax)] = \frac{1}{a} L\left(\frac{s}{a}\right)$.

Because $L[f(ax)] = \int_0^\infty e^{-sx}\, f(ax)dx = \int_0^\infty e^{-\frac{sy}{a}}\, f(y)dy/a$ [putting $ax = y \Rightarrow dx = \frac{dy}{a}$]

$L[f(ax)] = \frac{1}{a}\int_0^\infty e^{-sy/a}\, f(y)dy = \frac{1}{a} L\left(\frac{s}{a}\right)$. Thus we get $L\,[f(ax)] = \frac{1}{a} L\left(\frac{s}{a}\right)$

b) Shifting Property:

First Shifting Property: By this property if $L[f(x)] = L(s)$, then $L[e^{ax}f(x)] = L(s-a)$

Because we get

$$L[e^{ax}f(x)] = \int_0^\infty e^{-sx}\, e^{ax}\, f(x)dx = \int_0^\infty e^{-(s-a)x}\, f(x)dx = \int_0^\infty e^{-yx}\, f(x)dx$$

[when $y = s - a$]

But we have $(s) = \int_0^\infty f(x)e^{-sx}\,dx$. So we get

$L[e^{ax}f(x)] = \int_0^\infty f(x)e^{-yx}\,dx = L(y) = L(s-a)$ Thus we get $L[e^{ax}f(x)] = L(s-a)$

Second Shifting Property: By this property, if we define a new function $G(x)$ such that $G(x) = f(x-a)$ for $x > a$

And $G(x) = 0$ for $x < a$ then for $L[f(x)] = L(s)$, then $L[G(x)] = e^{-as}L(s)$

Because here we have

$$L[G(x)] = \int_0^\infty G(x)e^{-sx}\,dx = \int_0^a G(x)e^{-sx}\,dx + \int_a^\infty G(x)e^{-sx}\,dx$$

Thus

$$L[G(x)] = 0 + \int_a^\infty f(x-a)e^{-sx}\,dx = \int_0^\infty f(y)e^{-s(y+a)}dy = e^{-as}\int_0^\infty f(y)e^{-sy}\,dy = e^{-as}L(s)$$

This is second shifting property of Laplace's transformation

c) Laplace Transform of 1st and 2nd order Derivatives:

Now consider a function $y = f(x)$. Thus we get the 1st derivative of this function $\frac{dy}{dx} = f'(x)$.

Here we have Laplace transform of 1st derivative of this function

$$L[f'(x)] = sL[f(x)] - f(0), \quad \text{where} \quad L[f(x)] = L(s)$$

This is because

$$L[f'(x)] = \int_0^{\infty} e^{-sx}\frac{dy}{dx}dx = [e^{-sx}y]_0^{\infty} - \int_0^{\infty}(-se^{-sx})\, y\, dx \; = -y(0) + s\int_0^{\infty} e^{-sx}\, y\, dx$$

So we get $L[f'(x)] = -f(0) + s\int_0^{\infty} e^{-sx} f(x)\, dx = -f(0) + sL(s)$

i.e. $L[f'(x)] = sL[f(x)] - f(0)$

Similarly for Laplace's transform of the 2nd order derivative of the function $y = f(x)$, we get

$$L\left[\frac{d^2y}{dx^2}\right] = \int_0^{\infty} e^{-sx}\frac{d^2y}{dx^2}dx = [e^{-sx}\frac{dy}{dx}]_0^{\infty} + s\int_0^{\infty} e^{-sx}\frac{dy}{dx}dx = [e^{-sx}f'(x)]_0^{\infty} + s\int_0^{\infty} e^{-sx}\, f'(x)dx$$

$$L\left[\frac{d^2y}{dx^2}\right] = -f'(0) + sL[f'(x)]$$

But we have Laplace transform for the 1st derivative of the function $f(x)$

$L[f'(x)] = sL[f(x)] - f(0)$. Thus we get Laplace's transform of the 2nd order derivative of the function $y = f(x)$ as

$$L\left[\frac{d^2y}{dx^2}\right] = L[f''(x)] = -f'(0) + sL[f'(x)] = -f'(0) + s\{sL[f(x)] - f(0)\} \; .$$

So finally we get

$$L[f''(x)] = -f'(0) - sf(0) + s^2L(s) \quad \text{Or,} \quad L[f''(x)] = s^2L[f(x)] - sf(0) - f'(0)$$

d) Laplace transform of Integral of the function $f(x)$:

Here this transform is mathematically given by

$$L\left[\int_0^{x} f(x)dx\right] = \frac{1}{s}L(s), \text{where } L[f(x)] = L(s)$$

Because let us consider that $\phi(x) = \int_0^x f(x)dx$ and $\phi(0) = 0$. So we get $\phi'(x) = f(x)$

But for Laplace transform of the 1st derivative of the function $\phi(x)$ we have

$$L[\phi'(x)] = sL[\phi(x)] - \phi(0) = sL[\phi(x)]$$

[since we have considered the initial condition $\phi(0) = 0$]

So we get $L[\phi(x)] = \frac{1}{s} L[\phi'(x)] \Rightarrow L[\int_0^x f(x)dx] = \frac{1}{s} L[f(x)] = \frac{1}{s} L(s)$. This is Laplace transform for the integral of a function $f(x)$.

e) Convolution theorem for Laplace Transform:

For any two functions $f_1(x)$ and $f_2(x)$, the convolution of these two functions is also defined as $f_1(x) * f_2(x) \equiv \int_0^x f_1(y)\ f_2(x-y)dy$.

So if Laplace transforms $L[f_1(x)] = L_1(s)$ and $L[f_2(x)] = L_2(s)$, then by this convolution theorem $L[f_1(x) * f_2(x)] = L[\int_0^x f_1(y)\ f_2(x-y)dy] = L_1(s)L_2(s)$

To establish this theorem, we have

$$L[f_1(x) * f_2(x)] = L\left[\int_0^x f_1(y)\ f_2(x-y)dy\right] = \int_0^\infty e^{-sx} \int_0^x f_1(y)\ f_2(x-y)dy\, dx$$

$$= \int_0^\infty \int_0^x e^{-sx} f_1(y)\ f_2(x-y)dy\, dx .$$

Here the double integral being taken over the infinite region in the first quadrant lying between $y = 0$ and $y = x$. Changing the order of integration, we must take the limit from $x = y$ to $x = \infty$.

Thus we get $L[f_1(x) * f_2(x)] = \int_0^\infty \int_0^x e^{-sx} f_1(y)\ f_2(x-y)dy\, dx$

$$= \int_0^\infty \int_0^x e^{-sx} f_1(y)\ f_2(x-y)dxdy = \int_0^\infty e^{-sy}\ f_1(y)dy . \int_y^\infty e^{-s(x-y)}\ f_2(x-y)dx$$

$$= \int_0^\infty e^{-sy}\ f_1(y)dy \int_0^\infty e^{-sz}\ f_2(z)dz, \qquad [\text{ where } z = x - y]$$

$$= \int_0^\infty e^{-sx}\ f_1(x)dx \int_0^\infty e^{-sx}\ f_2(x)dx = L_1(s)L_2(s).$$

Thus $L[f_1(x) * f_2(x)] = L_1(s)L_2(s) \rightarrow$ this is convolution theorem for Laplace transformation.

f) Laplace's Integral Transform for Periodic Function:

Let $f(x)$ be a periodic function of period $p.$ Thus we have $f(x) = F(x+p)$ and then we have

$$L[f(x)] = \int_0^\infty e^{-sx}\ f(x)dx = \int_0^p e^{-sx}\ f(x)dx + \int_p^{2p} e^{-sx}\ f(x)dx + \int_{2p}^{3p} e^{-sx}\ f(x)dx + \cdots$$

$$= \int_0^p e^{-sy}\ f(y)dy + \int_0^p e^{-s(y+p)}\ f(y+p)dy + \int_0^p e^{-s(y+2p)}\ f(y+2p)dy + \cdots$$

$$= \int_0^p e^{-sy}\ f(y)dy + \int_0^p e^{-p(y+p)}\ f(y)du + \int_0^p e^{-p(y+2p)}\ f(y)dy + \cdots$$

$$= (1 + e^{-sp} + e^{-2sp} + \cdots)\int_0^p e^{-sy}\ f(y)dy = \frac{1}{1-e^{-sp}}\int_0^p e^{-sx}\ f(x)dx.$$

This is Laplace's transform for periodic function.

g) Laplace's Transform for Unit Step Function:

The unit step function $y(x-a)$ is defined as

$$y(x-a) = 0 \text{ when } x < a \text{ and } y(x-a) = 1 \text{ when } x \geq a \text{ where } a \geq 0$$

Thus Laplace transform of this step function is

$$L[y(x-a)] = \int_0^\infty e^{-sx}\ y(x-a)dx = \int_0^a e^{-sx}.0dx + \int_a^\infty e^{-sx}.1\,dx = 0 + \left|\frac{e^{-sx}}{-s}\right|_a^\infty = \frac{e^{-sa}}{s}$$

h) Laplace's Transform for Delta function:

This is given by $L[\delta(x-a)] = \int_0^\infty e^{-sx}\ \delta(x-a)dx = \int_0^\infty f(x)\,\delta(x-a)dx = f(a) = e^{-sa}$

i) Laplace Transformation of Gaussian Function:

$$L\left\{e^{-\alpha x^2}\right\} = L(s) = \int_0^\infty e^{-sx}\ e^{-\alpha x^2}\,dx = e^{s^2/4\alpha}\int_0^\infty e^{-\alpha\left(x^2+\frac{2sx}{2\alpha}+\frac{s^2}{4\alpha^2}\right)}\,dx$$

$$= e^{s^2/4\alpha}\int_0^\infty e^{-\alpha\left(x+\frac{s}{2\alpha}\right)^2}\,dx$$

Now substitute $\sqrt{\alpha}\left(x+\frac{s}{2\alpha}\right) = z$ Or, $dx = \frac{dz}{\sqrt{\alpha}}$

Thus we get $L\left\{e^{-\alpha x^2}\right\} = \frac{1}{\sqrt{\alpha}}\, e^{s^2/4\alpha}\int_{s/2\sqrt{\alpha}}^\infty e^{-z^2}\,dz$

But we know that the error function is given by $erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-y^2}dy$ and the complementary error function is given by $erfc(\sqrt{x}) = 1 - erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\int_{\sqrt{x}}^\infty e^{-y^2}dy$

So we get

$$L\left\{ e^{-\alpha x^2}\right\} = \frac{1}{\sqrt{\alpha}}\, e^{s^2/4\alpha} \int_{s/2\sqrt{\alpha}}^{\infty} e^{-z^2}\, dz = \frac{1}{\sqrt{\alpha}}\, e^{\frac{s^2}{4\alpha}} \frac{\sqrt{\pi}}{2} \operatorname{erfc}\left(\frac{s}{2\sqrt{\alpha}}\right)$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\; e^{\frac{s^2}{4\alpha}}. \operatorname{erfc}\left(\frac{s}{2\sqrt{\alpha}}\right)$$