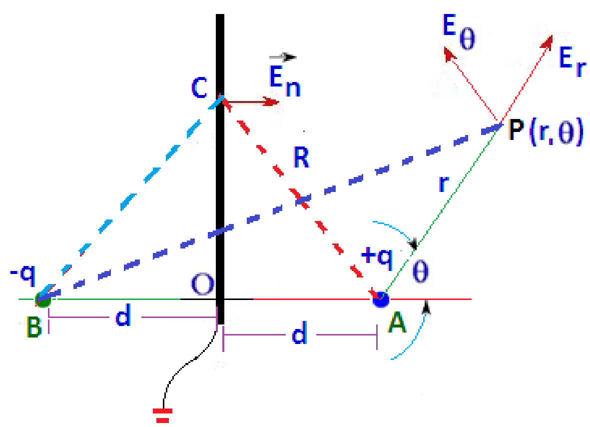
Electrostatics Image Problem:

Point Charge Placed in Front of an Earthed Conducting Plane:

Let a point charge $+q$ be placed at a distance d from an infinite conducting plane as shown in figure. We assume that the conducting plane is made grounded i.e., it is kept at zero potential. We want to calculate the potential and field intensity at any point P in neighbor region as shown for this whole electrostatic system. Here the source charge q induces opposite charge on the entire surface of that conducting plane and in this electrostatic image problem the specific conditions required to be obeyed are



(i) On the right-hand side of the plane (where the solution is desired), Laplace's equation will be satisfied everywhere about the point A where the source charge q is placed.

(ii) The potential at every point on the conducting plane must be zero because it is earthed.

(iii) The potential at infinity usually must be zero.

The symmetry of the problem suggests that the induced charges on the conducting plane may be replaced by an image charge $-q$ placed at any location where the solution is not desired, i.e., the location of this image charge can be taken on the left-hand side of the conducting plane. The position of the image charge is now taken at the point B as shown such that $BO = OA = d$. This placement of the image charge must satisfy all the above conditions. We can review these conditions ones again.

(i) The placement of the image charge $-q$ at B does not violate satisfaction of Laplace's equation on the right-hand side except at the point A.

(ii) To find potential of the conducting plane, we consider any point C on the plane where the potential at that point is $\phi_c = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{AC} - \frac{1}{BC}\right)$. The first term represents the potential due to $+q$ at A and the second term due to $-q$ at B. Since $AC = BC$, ϕ_c is zero as expected.

(iii) The location of the image charge must not affect the condition of the potential at infinity.

Thus the image charge $-q$ satisfies all the conditions so that the potential at P may be calculated using it instead of the actual induced charges. The name **"image charge"** for $-q$ arises from the optical analogy of mirror reflection.

For Potential and Field at P:

Let (x, y) be the Cartesian coordinates of the point P with respect to the origin O in the two dimensional diagram as shown in figure. The potential ϕ at P is obtained by summing the contributions due to the charge $+q$ at A and the charge $-q$ at B. Thus

$$\phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{AP} - \frac{1}{BP}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-d)^2+y^2}} - \frac{1}{\sqrt{(x+d)^2+y^2}}\right)$$

And the components of the electric field at P along x- and y-directions are given by $E_x = -\frac{\partial\phi}{\partial x}$ and $E_y = -\frac{\partial\phi}{\partial y}$

The expressions for E_x and E_y are readily obtained from potential expression as given above. We can also determine the potential and field at P using the polar coordinates (r, θ) with respect to the origin A, the position of the point charge $+q$.

We have $\phi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{\sqrt{r^2+4d^2+4rd\cos\theta}}\right)$. The radial component of the electric field $\vec{E}$ at P is given by $E_r = -\frac{\partial\phi}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{r+2d\cos\theta}{(r^2+4d^2+4rd\cos\theta)^{\frac{3}{2}}}\right]$. And similarly the θ - component or polar component of the electric field $\vec{E}$ is given by $E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{qd\sin\theta}{2\pi\varepsilon_0(r^2+4d^2+4rd\cos\theta)^{\frac{3}{2}}}$.

For Surface Charge Density of Induced Charge on the Conducting Plane:

Since the electric field at any point C on the conducting plane is normal to the plane, the component of the field normal to the surface at the point P is given by

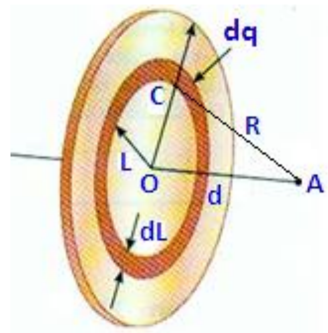
$$E_n = E_r\cos\theta + E_\theta\cos\left(\frac{\pi}{2}+\theta\right) = E_r\cos\theta - E_\theta\sin\theta = \frac{q}{4\pi\varepsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta+2d}{(r^2+4d^2+4rd\cos\theta)^{\frac{3}{2}}}\right].$$

At the point C, $r = R$ and $\cos\theta = -\frac{d}{R}$, where $R = AC$.

Therefore, the normal component of the field at C, obtained from above equation is $E_{nc} = -\frac{q}{4\pi\varepsilon_0}\frac{2d}{R^3} = -\frac{qd}{2\pi\varepsilon_0 R^3}$.

Thus surface density of charge at C is given by $\sigma = \varepsilon_0 E_{nc} = -\frac{qd}{2\pi R^3}$

This shows that the induced charge density is negative, i.e., opposite charge is induced on the conducting surface as expected. The induced charge density is a maximum at the point $\mathbf{O}$ and falls off as the cube of the distance of the position on the conducting surface from the charge $+\mathbf{q}$ at $\mathbf{A}$.



For total Induced Charge on the Conducting Surface:

We also have from figure $\mathbf{OC = (R^2 - d^2)^{1/2} = L}$ (say). The charge induced in a ring of radius $\mathbf{L}$ and thickness $\mathbf{dL}$ is $\mathbf{dQ = 2\pi L\sigma dL}$

Since we also get $\mathbf{L = RdR}$, the total induced charge $\mathbf{Q}$ on the plane is

$\mathbf{Q = 2\pi \int_d^\infty \sigma RdR = -qd \int_d^\infty \frac{dR}{R^2} = -q}$ .Thus the total induced charge is $-\mathbf{q}$ which is expected.