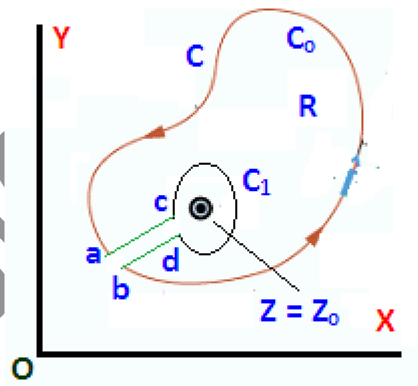
Cauchy's 2nd Integral Theorem:

This theorem states that if a function $\varphi(z) = \frac{f(z)}{z-z_o}$ is analytic at every point in a region except at a point $z = z_o$ in the region R as enclosed by a closed curve C then

$$f(z_o) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_o}dz \quad \text{i.e.} \quad \int_C \frac{f(z)}{z-z_o}dz = 2\pi i.f(z_o)$$

Because here for this function $\varphi(z) = \frac{f(z)}{z-z_o}$, it has singular point at $z = z_o$, we can now isolate this point $z = z_o$ by an infinitesimally small circle C_1 of radius r and centre at the point $z = z_o$ where $r \to 0$.

If we now connect these two close contours C and C_1 by a very narrow channel as shown in figure where $a \to b$ and $c \to d$ then we definitely get a close region as enclosed by the close contour C_o where



$$C_o = C(\text{anti}-\text{Clock}) + ac + C_1(\text{Clockwise}) + db$$

Since $\varphi(z)$ is analytic at each and every point in that close region as enclosed by the close contour C_o we have from Cauchy's 1st Integral theorem $\int_{C_o}\varphi(z)dz = 0$

But from figure

$$\int_{C_o}\varphi(z)dz = \int_{C_o}\frac{f(z)}{z-z_o}dz$$

$$= \int_{C(\text{anti}-\text{Clock})}\frac{f(z)}{z-z_o}dz + \int_a^c \frac{f(z)}{z-z_o}dz + \int_{C_1(\text{Clockwise})}\frac{f(z)}{z-z_o}dz + \int_d^b \frac{f(z)}{z-z_o}dz = 0$$

But at limiting conditions $a \to b$ and $c \to d$ (Since originally both C and C_1 are closed) we basically have $\int_d^b \frac{f(z)}{z-z_o}dz = -\int_b^d \frac{f(z)}{z-z_o}dz = -\int_a^c \frac{f(z)}{z-z_o}dz$

That is $\int_a^c \frac{f(z)}{z-z_o}dz + \int_d^b \frac{f(z)}{z-z_o}dz = 0$. Hence from above equation we finally get

$$\int_{C(\text{anti}-\text{Clock})}\frac{f(z)}{z-z_o}dz + \int_{C_1(\text{Clockwise})}\frac{f(z)}{z-z_o}dz = 0 \text{ ------- (1)}$$

But to find $\int_{C_1}\frac{f(z)}{z-z_o}dz$ we have the equation of the circular contour C_1 as $C_1: |z - z_o| = r$

Thus we have $z - z_o = re^{i\theta} \Rightarrow dz = ire^{i\theta}d\theta$. Thus we get

$$\int_{C_1}\frac{f(z)}{z-z_o}dz = \int_{C_1}\frac{f(z_o+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta = i\int_{C_1}f(z_o+re^{i\theta})\,d\theta \quad \text{where} \quad r \to 0$$

Hence $\int_{C_1} \frac{f(z)}{z-z_o} dz = i \int_{C_1} f(z_o)\, d\theta = i.f(z_o) \int_{C_1} d\theta = i.f(z_o) \int_0^{2\pi} d\theta = 2\pi i\, f(z_o)$

Thus we have from above equation (1)

$$\int_{C(anti-Clock)} \frac{f(z)}{z-z_o} dz = -\int_{C_1(Clockwise)} \frac{f(z)}{z-z_o} dz = \int_{C_1(anti-Clock)} \frac{f(z)}{z-z_o} dz = 2\pi i. f(z_o)$$

Thus the contour integration over C is just replaced by the contour integration over C_1 when both are taken in the same sense. Hence in general we get

$$\int_C \frac{f(z)}{z-z_o} dz = 2\pi i. f(z_o)$$

This is Cauchy's 2nd Integral Theorem.