Symmetries and Conservation Laws in Lagrangian Dynamics:

a) Conservation of Linear Momentum:

Here we now consider the translational motion of a conservative system and since q_j is the generalized coordinate, we take dq_j as generalized displacement of that translational motion. As we have for conservative system, the kinetic energy $T \neq T(q_j)$, we have $\left(\frac{\partial T}{\partial q_j}\right) = 0$. Again since the potential energy of that system $V \neq V(\dot{q}_j)$ we have $\frac{\partial V}{\partial \dot{q}_j} = 0$. Thus from Lagrange's equation we can write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial V}{\partial q_j} = 0 \Rightarrow \frac{d}{dt}(p_j) = -\frac{\partial V}{\partial q_j} \Rightarrow \dot{p}_j = -\frac{\partial V}{\partial q_j} = Q_j \rightarrow \text{Generalized Force}$$

But for this translational motion of the conservative system, since we have the generalized force $Q_j = \sum_{i=1}^{N} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_j}$, for unit vector $\hat{n}$ along the direction of the translational motion, we have $\delta\vec{r}_i = \hat{n}\delta q_j$ and in limiting case $\delta q_j \rightarrow 0$, we get $\partial\vec{r}_i = \hat{n}\partial q_j \Rightarrow \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n}$ and we get $Q_j = \sum_{i=1}^{N} \vec{F}_i . \hat{n} = \hat{n}. \sum_{i=1}^{N} \vec{F}_i = \hat{n}. \vec{F} \rightarrow$ component of total effective force along the direction of translational motion.

Again for this N particle system in translational motion, since the total kinetic energy of the system is $T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{r}_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{\vec{r}}_i . \dot{\vec{r}}_i)$, the corresponding generalized momentum is

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i . \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \vec{v}_i . \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} m_i \vec{v}_i . \hat{n} = \hat{n}. \sum_{i=1}^{N} m_i \vec{v}_i = \hat{n}. \vec{p} \rightarrow$$

Component of total linear momentum along the direction of translational motion

But since we have mentioned earlier that $\dot{p}_j = Q_j$, we get $\frac{d}{dt}(\hat{n}.\vec{p}) = \hat{n}.\vec{F}$ and this equation can easily be written as

$$\hat{n}.\frac{d}{dt}(\vec{p}) = \hat{n}.\vec{F} \Rightarrow \frac{d}{dt}(\vec{p}) = \vec{F} \quad \text{And obviously for } \vec{F} = 0 , \vec{p} = \text{Constant}$$

This is conservation of linear momentum as obtained in Lagrangian dynamics.

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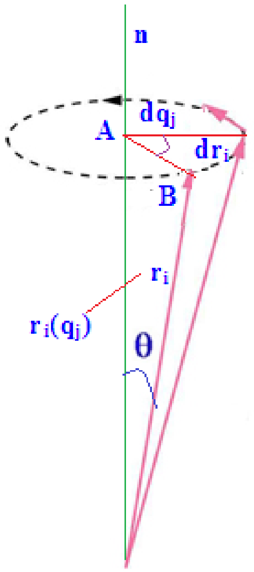
b) Conservation of Angular Momentum:

We consider the rotation of a system about an axis in circular path as shown, where the angular coordinate is taken as the generalized coordinate.



As we see from figure, the infinitesimal change of generalized coordinate q_j corresponds to an infinitesimal rotation of a vector $\vec{r}_i$ and in that case $|d\vec{r}_i| = ABdq_j = r_i \text{Sin}\theta . dq_j$

Here we can have $\left|\frac{\partial \vec{r}_i}{\partial q_j}\right| = r_i \text{Sin}\theta \Rightarrow \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$

With this result, we now have the generalized force

$$Q_j = \sum_{i=1}^{N} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} \vec{F}_i . (\hat{n} \times \vec{r}_i)$$

And this can be written as

$Q_j = \sum_{i=1}^{N} \hat{n}.(\vec{r}_i \times \vec{F}_i) = \sum_{i=1}^{N} \hat{n}.(\vec{\tau}_i) = \hat{n}.\sum_{i=1}^{N} \vec{\tau}_i = \hat{n}.\vec{\tau} \rightarrow$ Component of total torque along the axis of rotation

Again in this case the generalized momentum

$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i . \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i \vec{v}_i . \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} m_i \vec{v}_i . (\hat{n} \times \vec{r}_i) = \sum_{i=1}^{N} \hat{n}.(\vec{r}_i \times m_i \vec{v}_i) = \hat{n}.\sum_{i=1}^{N} \vec{L}_i = \hat{n}.\vec{L} \rightarrow$ Component of total angular momentum along the axis of rotation

Again we have from earlier discussion of conservative system $\dot{p}_j = Q_j$ and then we get $\frac{d}{dt}(\hat{n}.\vec{L}) = \hat{n}.\vec{\tau} \Rightarrow \hat{n}.\frac{d}{dt}(\vec{L}) = \hat{n}.\vec{\tau} \Rightarrow \frac{d}{dt}(\vec{L}) = \vec{\tau}$ and obviously for $\vec{\tau} = 0$, $\vec{L} =$ Constant

This is conservation of angular momentum in rotational dynamics as obtained in Lagrangian dynamics.

c) Conservation of Energy:

Here since for conservative system, kinetic energy is not function of generalized coordinates and potential energy not function of generalized velocity, we have for constrained motion of such conservative system with constraint of motion not function of time explicitly, the Lagrangian of the system $L = L(q_j, \dot{q}_j)$

In that case $dL = \sum_{j=1}^{f} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \Rightarrow \frac{dL}{dt} = \sum_{j=1}^{f} \frac{\partial L}{\partial q_j} . \frac{dq_j}{dt} + \sum_{j=1}^{f} \frac{\partial L}{\partial \dot{q}_j} . \frac{d\dot{q}_j}{dt}$

But we have also $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$. Hence we get $\frac{dL}{dt} = \sum_{j=1}^{f} \{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) . \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} . \frac{d\dot{q}_j}{dt}\}$

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Finally we get

$$\frac{dL}{dt}=\sum_{j=1}^{f}\frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right)=\sum_{j=1}^{f}\frac{d}{dt}\left(\dot{q}_j\frac{\partial T}{\partial \dot{q}_j}\right)=\sum_{j=1}^{f}\frac{d}{dt}\left(\dot{q}_j p_j\right)=\frac{d}{dt}[\sum_{j=1}^{f}\left(\dot{q}_j p_j\right)]$$

This gives $\frac{d}{dt}[L-\sum_{j=1}^{f}(\dot{q}_j p_j)]\ =0\ \Rightarrow\ L-\sum_{j=1}^{f}(\dot{q}_j p_j)\ =\text{Constant}$

$$\Rightarrow\sum_{j=1}^{f}\left(\dot{q}_j p_j\right)-L\ =\text{Constant}$$

But since the constraint of the system is not function of time explicitly i.e. ith position of the **N** particle system does not depend on time explicitly, we have the kinetic energy of that conservative system is a homogeneous function of generalized velocity in degree **2**, here we have $\sum_{j=1}^{f}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j}=2T\ \Rightarrow\sum_{j=1}^{f}(\dot{q}_j p_j)=2T$

Thus from above equation we get $2T-L=\text{Const.}\Rightarrow 2T-(T-V)=T+V=\text{Const.}\rightarrow$ this is energy conservation.

Basically the symmetry means invariance of certain property of the system under a given operation. More clearly if a function indicating the property of the system does not change under some operation then the system is said to have symmetry with respect to that operation.

As for example, consider the rotation of a cylinder about its own axis. For this rotation, the shape of that cylinder will not change. Thus the cylinder is said to have rotational symmetry about its own axis.

Similar to that we need to have a close system for which Lagrangian of the system does not change under operations – translation in space and rotation in space and also this Lagrangian will not depend on time explicitly i.e. $\frac{\partial L}{\partial t}=0$

Let us discuss our requirement.

a) If Lagrangian of the system does not depend on time explicitly then the constraints will also be independent of time explicitly and in that case as kinetic energy of the system will be homogeneous function of generalized velocity in degree 2, we have from our previous discussion, $\sum_{j=1}^{f}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j}=2T\ \Rightarrow\sum_{j=1}^{f}(\dot{q}_j p_j)=2T$ and since $\sum_{j=1}^{f}(\dot{q}_j p_j)-L\ =\text{Constant}$ we finally have

$2T-L=\text{Const.}\Rightarrow 2T-(T-V)=T+V=\text{Const.}\rightarrow$ this is energy conservation. For this energy conservation due to Lagrangian of the system independent of time explicitly, it is called Homogeneity of time

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b) Since the system is taken as a close system, the net effective external force on it will be zero and in that case the generalized force $Q_j = \sum_{i=1}^{N} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_j} = 0$. For this case conservation of linear momentum will occur and the Lagrangian of that close system will remain invariant in translational motion and system will have translational symmetry. This is called Homogeneity of space.

c) If Lagrangian of a close system remain invariant under small rotation of the coordinate frame with respect to any arbitrary axis of rotation then it is called isotropy of space and in this case total effective torque on rotating system will be zero and the related rotational coordinate will be cyclic in Lagrangian of the system. Such isotropy of space gives conservation of angular momentum.

These are the symmetries of a close system in respect to the invariance of Lagrangian of the system under relevant operations.

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