Planck's Law of Black Body Radiation:

The most elegant and straight-forward method for deducing Planck's radiation law is to consider that a chamber containing black radiation is full of photons in molecular chaos and determine how the energy is distributed amongst the photons.

In fact it was Planck's investigation on black radiation that gave the development of such theory. For such derivation, Planck's postulated that a black radiation chamber is filled up not only by radiation but by ideal gas molecules as well and also he introduced resonators of molecular dimensions as via media between radiation and gas molecules; the resonators absorb energy from radiation and transfer a part or whole of it to molecules during collision and the thermodynamic equilibrium is thereby established. The resonators introduced by Planck were electric dipoles having motion along the fixed axis, the centre of mass of each remaining stationary. The motion is supposed to be simple harmonic with a natural frequency ν.

Planck gave up the classical hypothesis of continuous emission of radiation by resonators and assumed that they emit energy only when the energy absorbed has a certain minimum value ϵ or some integral multiple of it.

Thus radiation of energy ϵ can be obtained from resonators having the energy content $\epsilon, 2\epsilon, 3\epsilon, \ldots\ldots r\epsilon, \ldots\ldots$ etc.

Let us now compute the mean energy of these resonators. Using Maxwell's classical formula, the probability that the resonator will possess the energy E is $e^{-E/kT}$. Therefore, the mean energy of a resonator $\bar{\epsilon}$ is given by

$$\bar{\epsilon} = \frac{\sum_o^\infty n\epsilon e^{-\frac{n\epsilon}{kT}}}{\sum_o^\infty e^{-\frac{n\epsilon}{kT}}} = \frac{\sum_o^\infty n\epsilon e^{-\beta n\epsilon}}{\sum_o^\infty e^{-\beta n\epsilon}} \left(\text{where } \beta = \frac{1}{kT}\right) \text{ and then}$$

$$\bar{\epsilon} = -\frac{d}{d\beta} \mathrm{In} \sum_o^\infty e^{-\beta n\epsilon} = -\frac{d}{d\beta} \mathrm{In} \frac{1}{1 - e^{-\beta\epsilon}}$$

Thus we get $\bar{\epsilon} = \frac{\epsilon e^{-\beta\epsilon}}{1 - e^{-\beta\epsilon}} = \frac{\epsilon}{e^{\beta\epsilon} - 1} = \frac{\epsilon}{e^{\epsilon/kT} - 1}$ and not kT as given by the equipartition law.

As by Planck's quantum idea $\epsilon = h\nu$, where h is Planck's constant, we thus get

$$\bar{\epsilon} = \frac{h\nu}{e^{h\nu/kT} - 1} \ldots\ldots\ldots (1)$$

But the number of modes of vibration per unit volume between the frequency ranges ν and $\nu + d\nu$ is, as already discussed, $n(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{c^3}$

[N.B: Because we should remember that from the idea of momentum space

$$N(k)dk = 2 \times \frac{4\pi k^2 dk}{\Delta k_x.\Delta k_y.\Delta k_z} \Longrightarrow N(\nu)d\nu = \frac{8\pi\left(\frac{2\pi\upsilon}{c}\right)^2 d\left(\frac{2\pi\upsilon}{c}\right)}{(\frac{2\pi}{L})^3} = \frac{8\pi\nu^2 d\nu}{c^3}.L^3 = \frac{8\pi\nu^2 d\nu}{c^3}.V$$

Here we multiply by the factor '2' as the electromagnetic radiation has two state of polarization.

Thus finally the number of modes of vibration per unit volume between the frequency ranges ν and $\nu + d\nu$ is $n(\nu)d\nu = \frac{N(\nu)d\nu}{V} = \frac{8\pi\nu^2 d\nu}{c^3}$]

So the energy density of radiation in the frequency interval ν and $\nu + d\nu$ is

$$u_\nu d\nu = [n(\nu)d\nu]\bar{\epsilon} = \frac{8\pi\nu^2 d\nu}{c^3} \times \frac{h\nu}{e^{h\nu/kT}-1} = \frac{8\pi h}{c^3}\frac{\nu^3}{(e^{h\nu/kT}-1)}d\nu \ldots\ldots\ldots(2)$$

which is the well-known Planck's law of radiation.

Expressed in wavelength by using, $\nu = \frac{c}{\lambda} \Rightarrow d\nu = \left|\frac{c}{\lambda^2}d\lambda\right|$, we obtain

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \times \frac{1}{e^{hc/\lambda kT} - 1}d\lambda \ldots\ldots\ldots(3)$$

This is an alternative form of Planck's law that gives energy density in the wavelength range λ and $\lambda + d\lambda$ in the spectrum of a black body.

Deduction of (a) Wien's Distribution Law, (b) Rayleigh-Jeans Law, (c) Stefan-Boltzmann Law and (d) Wien's Displacement law from Planck's law:

The different classical laws of radiation such as Wien's law, Rayleigh-Jean's law etc. conditionally follows Planck's law and thus all these classical laws can also be derived from Planck's semi classical law of Black body radiation and the discussions are given below.

a) Wien's Distribution Law:

In terms of wavelength λ, the Planck's distribution formula is $u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1}$

For short wavelength (or high frequency) and low temperature, λT is small so that the exponential term in the denominator has a value much greater than 1. Thus from the above equation

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT} - 1} = C_1\lambda^{-5}e^{-C_2/\lambda T}d\lambda$$

where the constants $C_1 = 8\pi hc$ and $C_2 = \frac{hc}{k}$.

This equation is Wien's radiation law, an essentially empirical formula containing two adjustable constants C_1 and C_2. Wien chose these constants so that the fit obtained by him was rather good except at long wavelengths.

b) Rayleigh-Jeans Law:

For long wavelength (or low frequency) and high temperature $\frac{hc}{\lambda kT} \ll 1$ Thus from above equation we have $u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\left(1+\frac{hc}{\lambda kT}+\cdots\right)-1} = \frac{8\pi kT}{\lambda^4}\, d\lambda$

which is Rayleigh-Jeans law. It shows that the energy density of radiation is inversely proportional to the fourth power of λ.

c) Wien's Displacement Law:

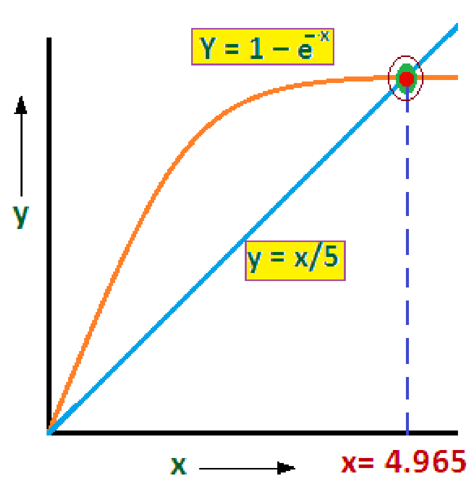
From Planck's distribution curve regarding black body radiation as we have discussed earlier, we see that for most abundant radiation at wavelength $\lambda = \lambda_m$, the energy density will be maximum. Thus in that case we mathematically have $\frac{\partial u_\lambda}{\partial \lambda}\Big]_{\lambda=\lambda_m} = 0$

Thus differentiating the right hand side of above equation for Planck's distribution formula with respect to λ and setting the result to zero, we obtain the λ (at temperature T) for which the energy density is maximum. Thus we have

$$-\frac{5\times 8\pi hc}{\lambda_m^6\left(e^{hc/\lambda_m kT}-1\right)} + \frac{8\pi hc}{\lambda_m^5}\cdot\frac{e^{hc/\lambda_m kT}}{\left(e^{hc/\lambda_m kT}-1\right)^2}\cdot\frac{hc}{kT\lambda_m^2} = 0.$$

Introducing $x = \frac{hc}{\lambda_m kT}$ and eliminating the common factors we get

$$\frac{xe^x}{e^x-1} = 5 \implies 1-e^{-x} = \frac{x}{5} = y$$



This transcendental equation that can be solved graphically or numerically shows at once that there must be a root in the neighbourhood of 5. Applying the usual method, the exact value is $x = 4.965$.

$$\text{Thus} \quad \frac{hc}{\lambda_m kT} = 4.965$$

$$\text{Or,} \quad \lambda_m T = \frac{hc}{4.965k} = 0.2896\ \text{cm.K} = \text{constant}$$

This is Wien's displacement law. It states that as the temperature of a black body is increased, the position of maximum emission moves in the direction of shorter waves in such a way that the product $\lambda_m T = $ constant. This provides us with a simple method of determining the temperature of the outer surface of all radiating body like sun.