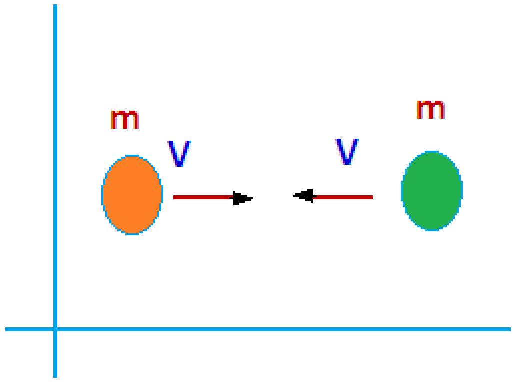
Variation of Mass with Velocity: Massless Particle:



Here we consider two relativistic frames S and S' where S frame is at rest and S' is moving with uniform velocity v. We now consider the collision of two equal masses m, approaching to each other with equal velocity V as observed by the S' observer.

If S observer notices this collision of two particles having respective masses m_1 and m_2, moving with respective velocity V_1 and V_2 before collision then we have from inverse velocity addition formula $V_1 = \frac{V+v}{1+\frac{Vv}{c^2}} \dots\dots\dots (1)$,

$V_2 = \frac{-V+v}{1-\frac{Vv}{c^2}} \dots\dots\dots (2)$

Thus with respect to S observer, we have from momentum conservation for collision of two particles $m_1V_1 + m_2V_2 = (m_1+m_2)v$ when total momentum is zero after collision in S' frame. Thus by using equation (1) and (2) we get

$$m_1\left(\frac{V+v}{1+\frac{Vv}{c^2}}\right) + m_2\left(\frac{-V+v}{1-\frac{Vv}{c^2}}\right) = (m_1+m_2)v \quad \text{Or,} \quad m_1\left[\frac{V+v}{1+\frac{Vv}{c^2}} - v\right] = m_2\left[v - \frac{-V+v}{1-\frac{Vv}{c^2}}\right]$$

$$\text{Or,} \quad m_1\left[\frac{V+v-v-\frac{Vv^2}{c^2}}{1+\frac{Vv}{c^2}}\right] = m_2\left[\frac{v-\frac{Vv^2}{c^2}+V-v}{1-\frac{Vv}{c^2}}\right] \quad \text{Or,} \quad \frac{m_1}{m_2} = \frac{1+\frac{Vv}{c^2}}{1-\frac{Vv}{c^2}} \dots\dots\dots (3)$$

Now we have

$$1-\frac{V_1^2}{c^2} = 1 - \frac{1}{c^2}\left(\frac{V+v}{1+\frac{Vv}{c^2}}\right)^2 = \frac{c^2\left(1+\frac{Vv}{c^2}\right)^2 - (V+v)^2}{c^2\left(1+\frac{Vv}{c^2}\right)^2}$$

$$= \frac{c^2+2Vv+\frac{V^2v^2}{c^2}-V^2-2Vv-v^2}{c^2\left(1+\frac{Vv}{c^2}\right)^2} = \frac{c^2\left(1-\frac{v^2}{c^2}\right)-V^2\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1+\frac{Vv}{c^2}\right)^2} = \frac{(c^2-V^2)\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1+\frac{Vv}{c^2}\right)^2}$$

Thus we have $$1-\frac{V_1^2}{c^2} = \frac{(c^2-V^2)\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1+\frac{Vv}{c^2}\right)^2} \dots\dots\dots (4)$$

Also we have $$1-\frac{V_2^2}{c^2} = 1-\frac{(-V+v)^2}{\left(1-\frac{Vv}{c^2}\right)^2}\frac{1}{c^2} = \frac{c^2\left(1-\frac{Vv}{c^2}\right)^2-(-V+v)^2}{c^2\left(1-\frac{Vv}{c^2}\right)^2}$$

$$= \frac{c^2\left[1-2\frac{vV}{c^2}+\frac{v^2V^2}{c^4}\right]-V^2-v^2+2vV}{c^2\left(1-\frac{Vv}{c^2}\right)^2} = \frac{c^2-2vV+\frac{v^2V^2}{c^2}-V^2-v^2+2vV}{c^2\left(1-\frac{Vv}{c^2}\right)^2} = \frac{c^2\left(1-\frac{v^2}{c^2}\right)-V^2\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1-\frac{Vv}{c^2}\right)^2}$$

Thus we get $$1-\frac{V_2^2}{c^2} = \frac{(c^2-V^2)\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1-\frac{Vv}{c^2}\right)^2} \dots\dots (5)$$

Dividing equation (4) by equation (5) we finally get

$$\frac{1-\frac{V_1^2}{c^2}}{1-\frac{V_2^2}{c^2}} = \frac{\left(1-\frac{vV}{c^2}\right)^2}{\left(1+\frac{vV}{c^2}\right)^2} \Rightarrow \frac{1-\frac{vV}{c^2}}{1+\frac{vV}{c^2}} = \frac{\sqrt{1-\frac{V_1^2}{c^2}}}{\sqrt{1-\frac{V_2^2}{c^2}}} \dots\dots (6)$$

From equations (3) and (6) we thus get $$\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{V_2^2}{c^2}}}{\sqrt{1-\frac{V_1^2}{c^2}}} \dots\dots\dots (7)$$

Let's consider that $V_1 = 0$, $m_1 = m_0$, $V_2 = V$ and $m_2 = m$. So from equation (7) we get $\frac{m_0}{m} = \sqrt{1-\frac{V^2}{c^2}} \Rightarrow m = \frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}} \dots\dots\dots (8)$

This is mass variation with velocity for a relativistic particle and here we see that for moving mass m and rest mass m_0, $m > m_0$ and moving mass m increases with increment of velocity of the particle.

Now for photon as a light particle since $V = c$, $m_0 = m\sqrt{1-\frac{c^2}{c^2}} = 0$ So photon is called massless particle and it has rest mass zero. Not only for photon but also Graviton is a massless particle, since Graviton is also moving with the same velocity of light in free space.

Mass-Energy Equivalency in Spatial Relativity:

For relativistic particle, its total energy is given by $E = KE + RE$ (Rest Energy) $= T + RE$ where RE is rest energy of the particle. Thus we have $T = E - RE \dots\dots\dots (1)$

Here from Work-Energy theorem we have kinetic energy for the relativistic particle having moving mass m while moving with velocity v and rest mass m_0 is given by

$$T = W = \int dW = \int Fdr = \int \frac{dp}{dt}\,.\,dr = \int dp\,\frac{dr}{dt} = \int v\,d(mv) = \int v\,d\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\,.\,v\right)$$

Thus we have $T = m_0 \int_{v=0}^{v=v} v\,d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = m_0\left[v\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right]_0^v - \int_0^v \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\,dv\Bigg]$

$$= m_0\left[\frac{v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{1}{2}c^2\int_0^v \frac{d\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right] = m_0\left[\frac{v^2}{\sqrt{1-\frac{v^2}{c^2}}} + c^2\left(\sqrt{1-\frac{v^2}{c^2}}\right)_0^v\right]$$

$$= m_0\left[\frac{v^2}{\sqrt{1-\frac{v^2}{c^2}}} + c^2\left(\sqrt{1-\frac{v^2}{c^2}} - 1\right)\right]$$

So we get $T = m_0\left[\left(\frac{v^2}{\sqrt{1-\frac{v^2}{c^2}}} + c^2\sqrt{1-\frac{v^2}{c^2}}\right) - c^2\right] = m_0\left(\frac{v^2+c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right) - m_0c^2$

Finally we get $T = mc^2 - m_0c^2 \ldots\ldots\ldots\ldots\ldots(2)$. This is mass energy equivalency of relativistic particle and here we see $E = mc^2 = T + m_0c^2$

Also we have from mass variation with velocity $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$

$$\Rightarrow m^2 - m^2v^2 = m_0^2c^2 \Rightarrow m^2c^4 = m^2v^2c^2 + m_0^2c^4 \Rightarrow (mc^2)^2 = p^2c^2 + m_0^2c^4$$

$$\Rightarrow E = \sqrt{p^2c^2 + m_0^2c^4}$$

So finally we have for mass-energy equivalency for any relativistic particle

$$E = mc^2 = T + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$$