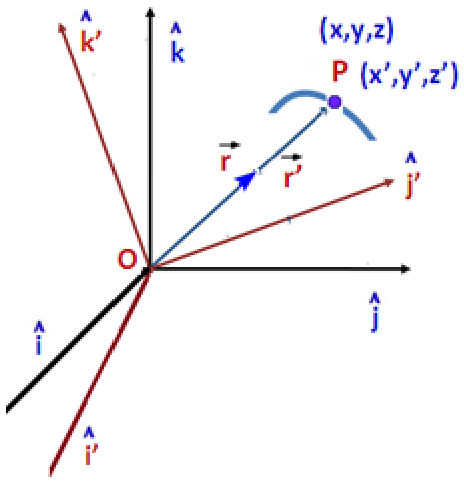
Uniformly Rotating Frame: Laws of Physics in Rotating Coordinate Systems:

Any rotating frame is an accelerated frame and that is why any rotating frame is itself a non inertial frame. Thus as shown in figure if we consider two frames **S** and **S'** be such that the frame **S** is at rest which is inertial frame and the other frame **S'** is rotating accelerated frame which is non inertial.

For the instantaneous position vectors $\vec{r}$ and $\vec{r'}$ of the same particles position **P** with respect to the frame **S** and **S'** respectively we have



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r'} = x'\hat{i'} + y'\hat{j'} + z'\hat{k'}$. And also $\vec{r} = \vec{r'} \implies \frac{d\vec{r}}{dt} = \frac{d\vec{r'}}{dt}$ But $[\frac{d\vec{r}}{dt}]_{Fixed} = [\frac{d\vec{r'}}{dt}]_{Fixed}$

$$= \frac{dx'}{dt}\hat{i'} + \frac{dy'}{dt}\hat{j'} + \frac{dz'}{dt}\hat{k'} + x'\frac{d\hat{i'}}{dt} + y'\frac{d\hat{j'}}{dt} + z'\frac{d\hat{k'}}{dt}$$

But we have

$[\frac{d\vec{r}}{dt}]_{Rot} = [\frac{d\vec{r'}}{dt}]_{Rot} = \frac{dx'}{dt}\hat{i'} + \frac{dy'}{dt}\hat{j'} + \frac{dz'}{dt}\hat{k'}$ .Hence we get $[\frac{d\vec{r'}}{dt}]_{Fixed} = [\frac{d\vec{r'}}{dt}]_{Rot} + x'\frac{d\hat{i'}}{dt} + y'\frac{d\hat{j'}}{dt} + z'\frac{d\hat{k'}}{dt}$.

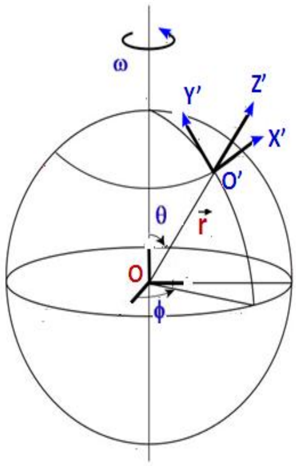
Also $\vec{v} = \vec{\omega} \times \vec{r} \implies \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \implies \frac{d}{dt} = \vec{\omega} \times$ So finally we get

$$[\frac{d\vec{r'}}{dt}]_{Fixed} = [\frac{d\vec{r'}}{dt}]_{Rot} + x'\frac{d\hat{i'}}{dt} + y'\frac{d\hat{j'}}{dt} + z'\frac{d\hat{k'}}{dt} = [\frac{d\vec{r'}}{dt}]_{Rot} + \vec{\omega} \times (x'\hat{i'} + y'\hat{j'} + z'\hat{k'})$$

That is $[\frac{d\vec{r'}}{dt}]_{Fixed} = [\frac{d\vec{r'}}{dt}]_{Rot} + \vec{\omega} \times \vec{r'} \implies [\frac{d}{dt}]_{Fixed} = [\frac{d}{dt}]_{Rot} + \vec{\omega} \times$

This is the relation between the time derivative operators in fixed frame and in rotating frame.

Development of Centrifugal Force and Coriolis Force in a Rotating Frame:



We take a rotating accelerated (non inertial) frame in which we take a particle moving with velocity $\vec{v}$. So if the motion of this particle be observed or be studied from both the frame **S** and **S'** where **S** is inertial and **S'** is non inertial frame then the acceleration of that moving particle as measured from inertial frame – S is given by

$$\vec{a} = (\frac{d^2\vec{r}}{dt^2})_{Fix} = [\frac{d}{dt}]_{Fixed}\{[\frac{d\vec{r}}{dt}]_{Fixed}\} = ([\frac{d}{dt}]_{Rot} + \vec{\omega}\times)([\frac{d\vec{r}}{dt}]_{Rot} + \vec{\omega}\times\vec{r})$$

So we have $\vec{a} = (\frac{d^2\vec{r}}{dt^2})_{Fix} = (\frac{d^2\vec{r}}{dt^2})_{Rot} + [\frac{d}{dt}]_{Rot}(\vec{\omega}\times\vec{r}) + \vec{\omega}\times[\frac{d\vec{r}}{dt}]_{Rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$ and finally acceleration $\vec{a} = (\frac{d^2\vec{r}}{dt^2})_{Fix} = (\frac{d^2\vec{r}}{dt^2})_{Rot} + \{[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\} + 2\vec{\omega}\times[\frac{d\vec{r}}{dt}]_{Rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$

Thus for particle's velocity in rotating frame $\vec{v} = [\vec{v}]_{Rot} = [\frac{d\vec{r}}{dt}]_{Rot}$ we have the acceleration

$$\vec{a} = (\frac{d^2\vec{r}}{dt^2})_{Rot} + \{[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\} + 2\vec{\omega}\times[\vec{v}]_{Rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= (\frac{d^2\vec{r}}{dt^2})_{Rot} + \{[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Multiplying both sides of this equation by the mass m of the particle we get

$$m(\frac{d^2\vec{r}}{dt^2})_{Fix} = m(\frac{d^2\vec{r}}{dt^2})_{Rot} + \{m[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\} + 2m(\vec{\omega}\times\vec{v}) + m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]$$

And this equation can be written as

$$\vec{F} = \vec{F}' + \{m[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\} + 2m(\vec{\omega}\times\vec{v}) + m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]$$

Or, $$\vec{F}' = \vec{F} + [-\{m[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\}] + [-2m(\vec{\omega}\times\vec{v})] + [-m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]]$$

Finally we have the effective force on the moving particle as observed from the rotating non inertial frame is $\vec{F}' = \vec{F} + [\vec{F}]_{Eular} + [\vec{F}]_{Coriolis} + [\vec{F}]_{Centrifugal}$ which is actually the resultant of the actual force acting on it and other three fictitious or pseudo forces developed in that moving particle in non inertial rotating frame. These three fictitious forces developed in a particle moving in rotating frame are respectively

$[\vec{F}]_{Euler} = -\{m[\frac{d\vec{\omega}}{dt}]_{Rot}\times\vec{r}\}$ → Euler force, $[\vec{F}]_{Coriolis} = -2m(\vec{\omega}\times\vec{v})$ → Coriolis force $\vec{F}]_{Centrifugal} = -m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]$ → Centrifugal force