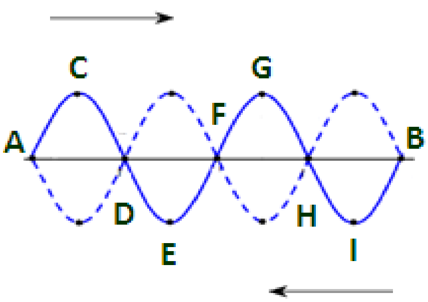
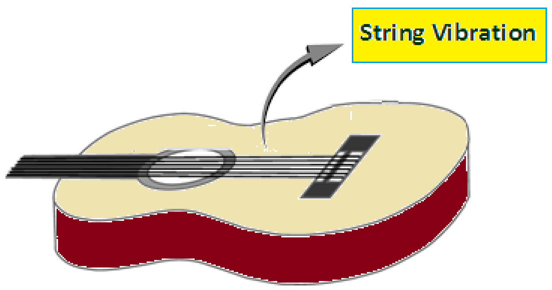
## Standing (Stationary) Waves in a String: Analytical Treatment:

When two progressive waves of same amplitude, wavelength and frequency approaches to each other after coming from opposite sense and superimposes with each other in opposite phase then the resultant wave motion produced on superposition will be confined within a certain region of that medium. This resultant wave is called stationary wave or standing wave. The basic characteristics of this standing or stationary wave are



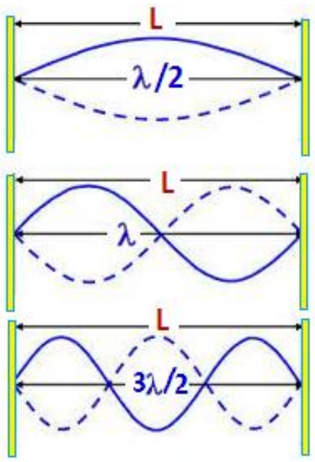

a) Standing wave cannot be expressed by the equation of a single progressive wave




b) At some few points on this standing wave, there occurs no vibration and then those points are called nodes where as there exists another few points on that wave motion at which the amplitude of vibration of the medium particle or medium layers is maximum and then those points are called antinodes

c) For standing wave nodes and antinodes appears in alternate manner and the separation between one node position from its neighbour antinodes is $\frac{\lambda}{4}$.

We now consider a wire of uniform cross section which is stretched horizontally at two ends. The length of the wire is $l$ and the tension of the wire is $T$ where as the mass per unit length is $m$.




Now if the wire is allowed to vibrate in transverse sense then it can be assumed that a transverse wave is generated along the length of the wire which reflects back from one end and superimposes with the incoming wave in opposite phase and finally a transverse standing wave will generated along the length of the string.

For this transverse vibration of string if the string vibrates with fundamental frequency then only one anti node will appear at the midpoint of the string with two nodes at the end points. Hence if $n_1$ be the frequency of fundamental tone then we have from figure the length of the string is given by $l = \frac{\lambda}{4} + \frac{\lambda}{4} = \frac{\lambda}{2} \Rightarrow \lambda = 2l$ and then $n_1 = \frac{v}{\lambda} = \frac{v}{2l}$.

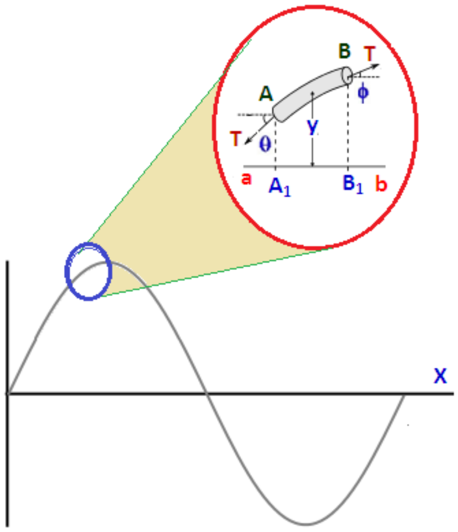
Again it can be shown mathematically that the velocity of transverse wave due to string vibration is given by $v = \sqrt{\frac{T}{m}}$ . Hence the fundamental frequency of the string vibration will be $n_1 = \frac{1}{2l}\sqrt{\frac{T}{m}}$. From this equation we can say that $n_1 \propto \frac{1}{l}$ when T and m are constants. Again $n_1 \propto \sqrt{T}$ when $m$ and $l$ are constants.

Also again we have $n_1 \propto \frac{1}{\sqrt{m}}$ for $T$ and $l$ constants. These are known as laws of transverse vibration of string. Now if the string be vibrated into two loops with an additional node position at the midpoint of the string then the tone appears is called second overtones.

If $n_2$ be the frequency of second overtone then since we have $l = \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda \Rightarrow \lambda = l$ we get the frequency of the second overtone $n_2 = \frac{v}{\lambda} = \frac{v}{l} = 2\frac{v}{2l} = 2n_1$. Similarly for 3rd overtone we get $n_2 = 3n_1$. Thus for p'th overtone the frequency will be

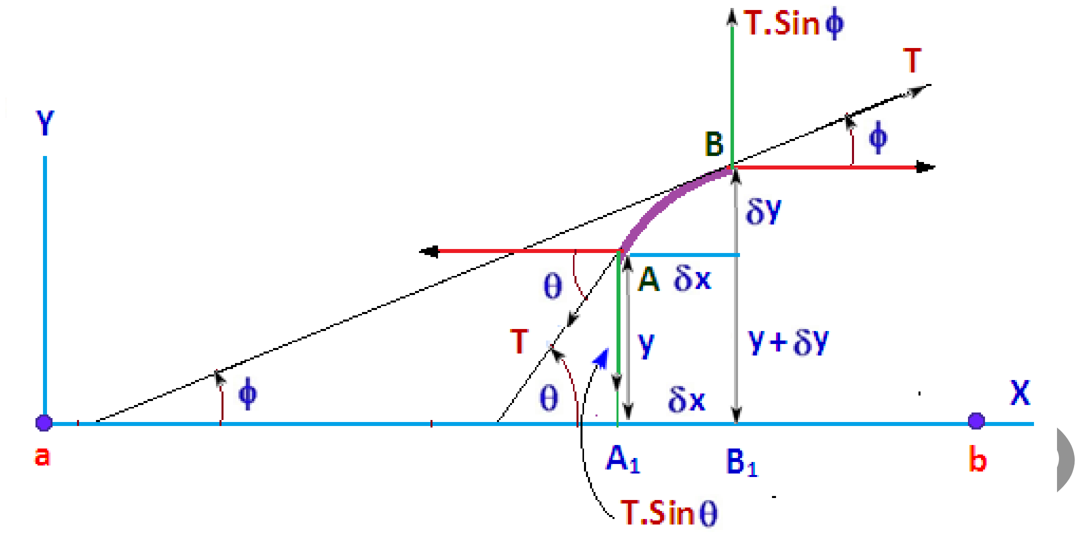
$$n_p = pn_1 = p.\frac{1}{2l}\sqrt{\frac{T}{m}}.$$

Thus we see that all the overtones for the transverse vibration of string are harmonics.




Let a string stretched between two fixed supports lie at rest along the $x-$axis. When a portion of the string is drawn along the $y-$axis , i.e. perpendicular to its length such that the amplitude of vibration is small, the tension of the string can be taken to be a constant in time as transverse waves proceed along the string.

Suppose that $ab$ is the stretched string in its position of rest, and $AB(= \delta l)$ is a small segment of the string in the displaced position. The string being perfectly flexible, the tension $T$ can be taken to be the same along the length of the string, acting tangentially at every point.




If $\theta$ is the angle made by the tangent at $A$ with $ab$, the transverse component of the tension will be $T_v = T\sin\theta$. Since $\theta$ is small $\sin\theta \simeq \tan\theta = \frac{\partial y}{\partial x}$.

Hence $T_v = T\frac{\partial y}{\partial x}$ Again vertical component of the tension at $B$ is

$$T_v = T\sin\varphi = T\frac{\partial}{\partial x}(y+\delta y) = T\frac{\partial}{\partial x}\left(y+\frac{\partial y}{\partial x}\delta x\right)$$

So the resultant force on the element $AB$ along the $y$ –direction is

$$T\frac{\partial}{\partial x}\left(y+\frac{\partial y}{\partial x}\delta x\right) - T\frac{\partial y}{\partial x} = T\frac{\partial^2 y}{\partial x^2}\delta x.$$

From Newton's second law of motion this force must be equated to the product of the mass of the element $AB$ and its acceleration. If $m$ is the mass per unit length of the string, the mass of the element $AB$ is $m\delta x$ and its acceleration is $\frac{\partial^2 y}{\partial t^2}$

So $T\frac{\partial^2 y}{\partial x^2}\delta x = m\delta x\frac{\partial^2 y}{\partial t^2}$ Or, $\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2} = v^2\frac{\partial^2 y}{\partial x^2}$. Thus we see that for such transverse wave propagation the velocity of transverse wave propagation is $v = \sqrt{\frac{T}{m}}$.