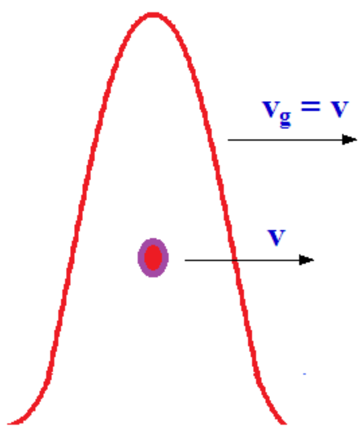
Wave Aspect of Matter

1. Basic Concept:

In the year 1901, the scientist Max Planck gave the concept of old quantum theory which is actually based on particle aspect of radiation. By this theory any monochromatic electromagnetic radiation can be considered as a stream of a number of discrete energy packet each containing finite amount of energy $h\nu$ where h is Planck's constant $(h = 6.626 \times 10^{-34}\ \mathrm{J.sec})$ and ν is the frequency of that monochromatic radiation. This energy packet is called photon or quanta and thus if the given radiation contains n number of such discrete monochromatic photon then the total energy of that radiation will be $E = nh\nu$

After that Planck established this quantum concept theoretically by explaining the characteristics of Black body radiation with the help of this concept. In the year 1905, Einstein gave the quantum explanation of photo electric effect by the help of this theory given by Planck.



Again in 1921 the scientist Compton gave the explanation of Compton scattering by this quantum theory. Also the scientist Dirac gave the explanation of Pair production with the help of this quantum theory. By this manner Plank's quantum theory was well established.

After a few years, in 1928 the great philosopher de Broglie gave the wave aspect of particle motion which is known as wave particle duality. By this aspect , any particle motion can be replaced the motion of matter wave which is basically the wave packet and according to de Broglie the wave length of that wave packet is inversely proportional to the momentum of the corresponding particle motion and it is given by $\lambda = \frac{h}{p} = \frac{h}{mv}$

The basic characteristics of this matter wave are

a) It is not a single continuous wave extending from $-\infty$ to $+\infty$, it is actually a wave packet which can be obtained by the superposition of a number of single wave having slightly varying frequency and wavelength.

b) The width of the wave packet is the estimation of the position uncertainty (Δx) and thus this wave particle duality is supported by Heisenberg's uncertainty principle.

c) The wave packet or the matter wave is a localized wave which can theoretically constructed by Fourier analysis and it moves with group velocity which must be equal to the particle velocity as required for perfect replacement.

d) The concept of wave packet motion as a representative of particle motion is also supported by Bohr's quantization of angular momentum which was the basis of Bohr's atomic structure. Thus this wave particle duality is supported by Bohr's atomic theory.

2. Support from Bohr's Atomic Theory:

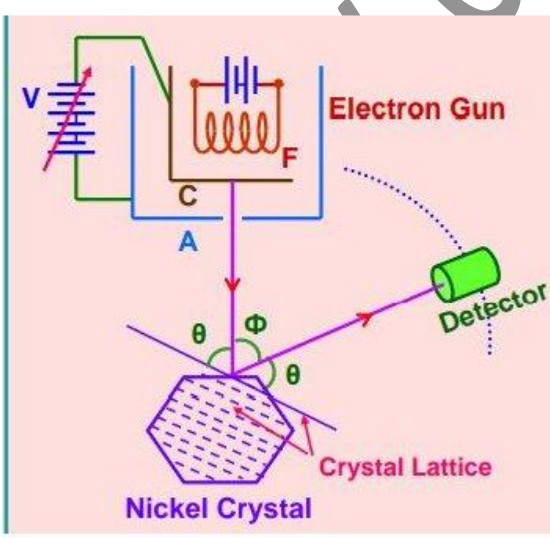
Consider the rotation of electron around nucleus in circular orbit as considered in Bohr's atom model. The electronic motion can be replaced by the motion wave packet as predicted by de Broglie and thus if we consider that for rotation of electron in $\mathbf{n}$ th circular orbit , the circumference of that orbit is covered by n number of such wave packet having wave length $\boldsymbol{\lambda}$ then the circumference of that nth orbit will be

$$2\pi r = n\lambda = n\frac{h}{p} = \frac{nh}{mv} \Rightarrow mvr = n\left(\frac{h}{2\pi}\right)$$

This is Bohr's quantization of angular momentum. Thus the concept of wave particle duality is consistent with Bohr's atom model.

3. Davison Germar Experiment:

The scientist Davison and Germar verified wave particle duality experimentally. In their experiment, they have used the electron beam coming from electron gun when the beam is made accelerated by **54 Volt** potential drops. After that they allowed that beam to incident on a **Ni** crystal and get diffraction pattern on the screen when placed in front of that crystal.



As the property of diffraction is a characteristic of wave, that electron beam is supposed to be wave motion equivalently. Here theoretically by wave particle duality, the equivalent wave length of electron beam can be obtained by the following manner.

If an electron be accelerated by the potential drop $\mathbf{V}$ then energetically we have

$$\frac{1}{2}mv^2 = eV \Rightarrow v = \sqrt{\frac{2eV}{m}} \Rightarrow mv = \sqrt{2meV}\,.$$

We get $$\lambda]_{electron} = \frac{h}{mv} = \frac{h}{\sqrt{2meV}} = \frac{h}{\sqrt{2me}} \cdot \frac{1}{\sqrt{V}} = \frac{12.26}{\sqrt{V}}\ A$$

Since in Davison Jermar's Experiment the electron beam is accelerated by 54 Volts potential drop, de Broglie wave length of this electron beam should be theoretically

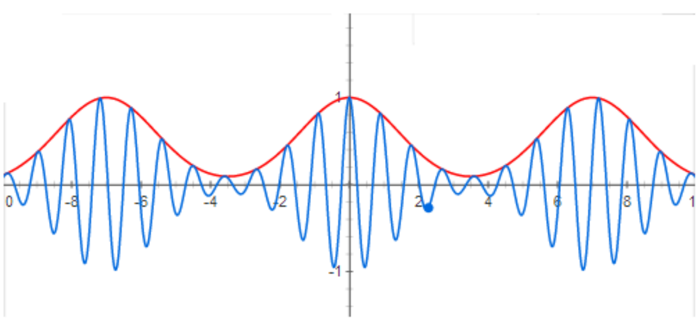
$$\lambda]_{electron} = \frac{12.26}{\sqrt{54}}\ A = 1.66\ A$$

In experiment of Davison Germar, they observed that 1st order diffraction maxima is obtained at $\varphi = 50^o$ angular position of the detector.

Here this angle φ is basically the angle between the incoming and reflected or scattered electron beams as shown in figure. Here the scattering of electron beam from the Nickel crystal is considered as the reflection of the electron beam from the crystal plane. Hence from this experimental observation, we have from Bragg's law of diffraction $2dSin\theta = n\lambda$ where we have from figure $2\theta = \pi - \varphi$ and then $\theta = 90 - 25 = 65$. Also for Ni Crystal d = 0.91A. Hence we get $2 \times 0.91 \times Sin65 = 1 \times \lambda \Rightarrow \lambda = 1.656\ A$

Thus the experimentally obtained value of de Broglie wave length of that electron beam is identical to the theoretical value of the same as obtained from wave particle duality. Hence wave particle duality is confirmed experimentally.

4. Group Velocity of Wave Packet:



When at least two or more than two waves of slightly different frequency and wavelength superimposes with each other then the amplitude of the resultant wave obtained will propagate in wave manner. The phase velocity of that amplitude wave is called group velocity in wave motion. Let us now consider the superposition of two waves having respective frequency ω_1 and ω_2 and propagation constant k_1 and k_2 . Thus by the principle of superposition the resultant wave will be

$$y = y_1 + y_2 = ASin(\omega_1 t - k_1 x) + ASin(\omega_2 t - k_2 x)$$

$$= 2A\,\text{Sin}\left(\frac{\omega_1+\omega_2}{2}t-\frac{k_1+k_2}{2}x\right)\text{Cos}\left(\frac{\omega_1-\omega_2}{2}t-\frac{k_1-k_2}{2}x\right)$$

$$= A_o\,\text{Sin}\left(\frac{\omega_1+\omega_2}{2}t-\frac{k_1+k_2}{2}x\right)$$

$$\Longrightarrow\ y = A_o\text{Sin}(\omega t-kx)\quad \text{for } \omega_1\approx\omega_2\approx\omega \text{ and } k_1\approx k_2\approx k$$

Here the amplitude wave of resultant motion will be

$$A_o = 2A\,\text{Cos}\left(\frac{\omega_1-\omega_2}{2}t-\frac{k_1-k_2}{2}x\right) = 2A\,\text{Cos}\left(\frac{\Delta\omega}{2}t-\frac{\Delta k}{2}x\right)$$

And the phase velocity of this amplitude wave will be $v_g = \frac{\frac{\Delta\omega}{2}}{\frac{\Delta k}{2}} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$.

This is the group velocity of wave motion. The velocity of that so called wave packet is equal to this group velocity as predicted in wave particle duality.

5. Equalization of Group Velocity of Wave Packet with the Velocity of Particle Motion:

Since wave particle duality is a general aspect of particle motion but it is ordinarily acceptable for the motion of micro particle. Basically the micro particles are relativistic particles we have from special relativity, the moving mass of the particle in relativity is

$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}$ where v is the velocity of the moving particle having mass m .

Again we have the group velocity of wave packet $v_g = \frac{d\omega}{dk} = \frac{d(2\pi\nu)}{d\left(\frac{2\pi}{\lambda}\right)} = \frac{d(\nu)}{d\left(\frac{1}{\lambda}\right)} = \frac{d(\nu)}{dv}\Big/\frac{d\left(\frac{1}{\lambda}\right)}{dv}$

Since $\nu = \frac{h\nu}{h} = \frac{mc^2}{h} = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}.\frac{c^2}{h} \Longrightarrow \frac{d(\nu)}{dv} = \frac{m_oc^2}{h}.\frac{d}{dv}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{m_oc^2}{h}.\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2v}{c^2}\right)$

$$\Longrightarrow \frac{d(\nu)}{dv} = \frac{m_o v}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

And $\frac{1}{\lambda}=\frac{mv}{h}=\frac{m_o v}{h\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{d\left(\frac{1}{\lambda}\right)}{dv}=\frac{m_o v}{h}.\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(\frac{-2v}{c^2}\right)+\frac{m_o}{h}.\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

$$\Rightarrow \frac{d\left(\frac{1}{\lambda}\right)}{dv}=\frac{m_o}{h}.\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left\{\frac{v^2}{c^2}+\left(1-\frac{v^2}{c^2}\right)\right\} \Rightarrow \frac{d\left(\frac{1}{\lambda}\right)}{dv}=\frac{m_o}{h}.\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

Then $v_g=\frac{d\omega}{dk}=\frac{d(\nu)}{dv}\Big/\frac{d\left(\frac{1}{\lambda}\right)}{dv}=\frac{m_o v}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\Big/\frac{m_o}{h}.\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}=v$

Hence the group velocity of this wave packet must be equal to the velocity of the particle motion. Also if we consider non relativistic particle then since we have $v_g=\frac{d\omega}{dk}=\frac{d\left(\frac{h}{2\pi}\omega\right)}{d\left(\frac{h}{2\pi}k\right)}=\frac{dE}{dp}$ we can now write down $v_g=\frac{d\omega}{dk}=\frac{dE}{dp}=\frac{p}{m}=\frac{mv}{m}=v=$ Particle velocity

Solved Problems

1. Find the typical de Broglie wavelength associated with a He atom in He gas at room temperature ($27^\circ C$) and 1 atm pressure; and compare it with the mean separation between two atoms under these conditions Given: Boltzmann constant$=1.38\times10^{-23}\ JK^{-1}$.

Ans: Given: $T=27+273=300\ K$, Mass of a He atom $=\frac{4\times10^{-3}}{6\times10^{23}}=\frac{2}{3}\times10^{-26}\ kg.$

From $E_k=\frac{1}{2}mv^2=\frac{3}{2}KT$ we get $p=\sqrt{2mE_k}=\sqrt{3mKT}$

Also $\lambda=\frac{h}{p}=\frac{h}{\sqrt{3mKT}}=\frac{6.63\times10^{-34}}{\sqrt{3\times\frac{2}{3}\times10^{-26}\times1.38\times10^{-23}\times300}}=0.73\times10^{-10}m.$

Again from $PV=RT=kNT$ (for 1 mole) and $\frac{V}{N}=\frac{KT}{P}$, But V/N is volume of 1 atom $(=r^3)$

And $r=\left(\frac{KT}{P}\right)^{1/3}=\left(\frac{1.38\times10^{-23}\times300}{1.013\times10^5}\right)^{1/3}$ or $r=34\times10^{-10}m$

On comparison, we find that $r\gg\lambda$.

2. Calculate de Broglie wave length of the electron orbiting in the n = 2 state of hydrogen atom.

Ans: Total energy of the electron in nth state $= \frac{-13.6}{n^2}\,\mathrm{eV}$

K.E of the electron in nth state $= \frac{13.6}{n^2}\,\mathrm{eV}$

K.E of the electron in 2nd state $= \frac{13.6}{2^2} = 3.4\ \mathrm{eV}, E_K = 3.4 \times 1.6 \times 10^{-19}\ \mathrm{J}$

$$\text{Now}\ \lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE_K}} = \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 3.4 \times 1.6 \times 10^{-19}}} = \frac{6.63 \times 10^{-34}}{9.95 \times 10^{-25}}$$
$$= 0.666 \times 10^{-9}\ \mathrm{m} = 0.666\ \mathrm{nm}$$

3. Compare the energy of an electron of de Broglie wavelength $1\ \mathrm{\AA}$ with that an X-ray photon of the same wavelength. $(h = 6.6 \times 10^{-34}\ \mathrm{Js})$

Ans: For the electron, $\lambda = \frac{h}{p}$ Or $p = \frac{h}{\lambda}$ Thus K.E. of the electron $E_K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$

Energy of X-ray photon, $E_X = h\nu = \frac{hc}{\lambda}$ and $\frac{E_K}{E_X} = \frac{h^2}{2m\lambda^2} \times \frac{hc}{\lambda} = \frac{6.6\times10^{-34}}{2\times9.1\times10^{-31}\times3\times10^{8}\times10^{-10}} = \frac{11}{910}$

4. Obtain de Broglie wavelength associated with thermal neutrons at room temperature $(27^\circ\mathrm{C})$. Hence, explain why a fast neutron beam needs to be thermalized with the environment before it can be used for neutron diffraction experiments. (Given: Boltzmann constant $(k) = 1.38 \times 10^{-23}$)

Ans: Given: $T = 27^\circ\mathrm{C} = 300\ \mathrm{K}$ Average kinetic energy of neutron at temp T

$$E_K = \frac{1}{2}mv^2 = \frac{3}{2}KT = \frac{3}{2} \times 1.38 \times 10^{-23} \times 300 = 6.21 \times 10^{-21}\ \mathrm{J}$$

$\lambda = \frac{h}{\sqrt{2m.E_K}} = \frac{6.63\times10^{-34}}{\sqrt{2\times1.675\times10^{-27}\times6.21\times10^{-21}}} = 1.45 \times 10^{-10} = 1.45\ \mathrm{\AA}$. As this wavelength is of the order of interatomic separation $(\sim 1\ \mathrm{\AA})$ in a crystal, so thermal neutrons are suitable for diffraction experiments.

5. X-rays of wavelength $0.82\ \mathrm{\AA}$ falls on a metal plate. Find the de Broglie wavelength associated with photoelectron emitted. Neglect work function of the metal. $(\text{Given}: h = 6.6 \times 10^{-34}\ \mathrm{Js}\ \text{and}\ c = 3 \times 10^{8}\mathrm{ms}^{-1})$

Ans: Given: $\lambda = 0.82 \times 10^{-10}\mathrm{m}, \phi = 0$ From $E_K = h\nu - \phi = \frac{hc}{\lambda} - 0 = \frac{hc}{\lambda}$

$$\lambda = \frac{h}{\sqrt{2m.E_K}} = \frac{h}{\sqrt{2m.\frac{hc}{\lambda}}} = \sqrt{\frac{h\lambda}{2mc}}, \lambda = \sqrt{\frac{6.6 \times 10^{-34} \times 0.82 \times 10^{-10}}{2 \times 9.1 \times 10^{-31} \times 3 \times 10^{8}}} = 0.099 \times 10^{-10} m$$
$$= 0.099\ \text{Å}$$

6. An alpha particle and a proton have their masses in the ratio of 4 : 1 and charges in the ratio of 2 : 1. Find the ratio of de Broglie wavelength associated with them when both (i) move with equal velocity (ii) have equal momentum (iii) have equal kinetic energy (iv) are accelerated through the same potential

Ans: Given: $\frac{m_\alpha}{m_p} = \frac{4}{1}$ and $\frac{Q_\alpha}{Q_p} = \frac{2}{1}$ (i) From $\lambda = \frac{h}{mv}, \frac{\lambda_\alpha}{\lambda_p} = \frac{m_p v}{m_\alpha v} = \frac{m_p}{m_\alpha} = \frac{1}{4}$

(ii) From $\lambda = \frac{h}{p}$. As momentum 'p' of both is equal, $\frac{\lambda_\alpha}{\lambda_p} = 1$

(iii) $\lambda = \frac{h}{\sqrt{2m.E_K}}$ and $\frac{\lambda_\alpha}{\lambda_p} = \sqrt{\frac{2m_p E_K}{2m_\alpha E_K}} = \sqrt{\frac{m_p}{m_\alpha}} = \frac{1}{2}$

(iv) From $\lambda = \frac{h}{\sqrt{2mqV}}$ we get $\frac{\lambda_\alpha}{\lambda_p} = \sqrt{\frac{m_p}{m_\alpha} . \frac{q_p}{q_a}} = \sqrt{\frac{1}{4} \times \frac{1}{2}} = \frac{1}{\sqrt{8}}$

7. A particle is moving three times as fast as an electron. The ratio of de Broglie wavelength of the particle to that of the electron is 1.813×10^{-4}. Calculate the particle's mass and identity the particle.

Ans: For the electron: $\lambda_e = \frac{h}{m_e v_e}$ For the particle, $\lambda_p = \frac{h}{m_p v_p}$ and $\frac{\lambda_p}{\lambda_e} = \frac{m_e}{m_p} \times \frac{v_e}{v_p} = \frac{1}{3}\frac{m_e}{m_p}$

Thus $m_p = \frac{1}{3} m_e \times \frac{\lambda_e}{\lambda_p} = \frac{1}{3} \times 9.1 \times 10^{-31} \times \frac{1}{1.813 \times 10^{-4}} = 1.675 \times 10^{-27} kg$. This particle could be a proton ot a neutron.