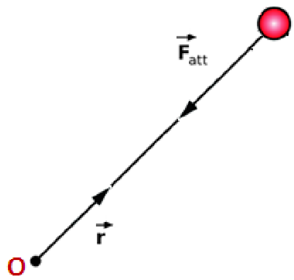
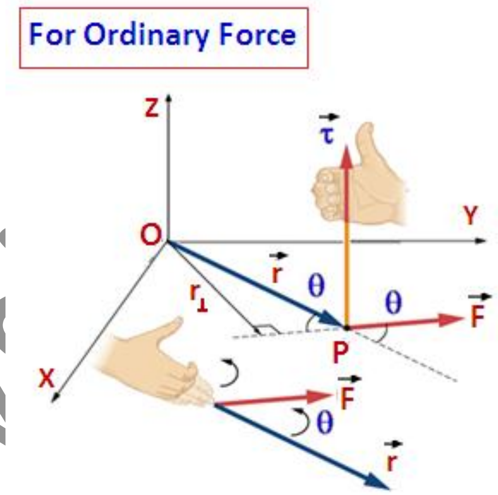
## Motion of a Particle under a Central Force Field: Energy and Angular Momentum Conservation under Central Force:

A central force is a particular force field which is always directed towards or away from a fixed point. That fixed point is called '**pole**' of the interaction and thus such central force may be **attractive** (i.e. **negative**) or **repulsive** (i.e. **positive**), which is always directed in the **radial sense.** The magnitude of this force solely depends on the distance $\mathbf{r}$ of the particle on which the force acts from the fixed point. Let us suppose that the force on a particle of mass $\mathbf{m}$ is $\vec{\mathbf{F}} = \mathbf{m}\,\vec{\mathbf{f}}(\mathbf{r})$. In that case we always have for central force $\vec{\mathbf{F}}$ ,

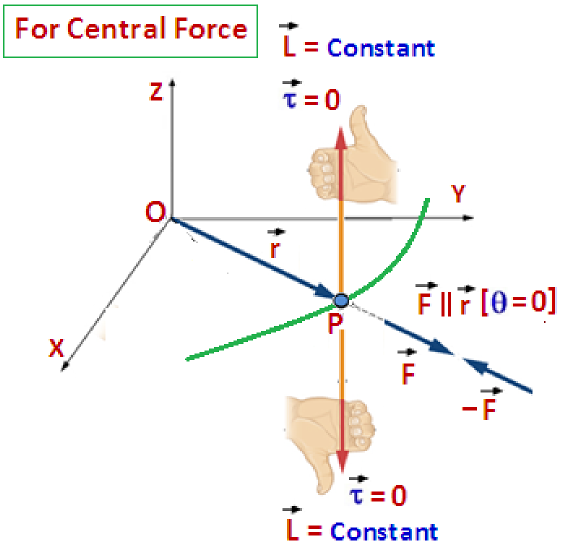



$$|\vec{\mathbf{F}}| = \mathbf{F} = \mathbf{m}\mathbf{f}(\mathbf{r}) = \mathbf{m}|\vec{\mathbf{f}}(\mathbf{r})| \quad \text{and also}$$
$$\vec{\mathbf{F}} = \mathbf{m}\,\vec{\mathbf{f}}(\mathbf{r}) = |\vec{\mathbf{F}}|(\pm\hat{\mathbf{r}}) = \pm\mathbf{m}\mathbf{f}(\mathbf{r}).\hat{\mathbf{r}}$$




With this basic criteria of this central force we should immediately have

**a)** $\vec{\mathbf{f}}(\mathbf{r}) = \frac{\mathbf{d^2\vec{r}}}{\mathbf{dt^2}} = \frac{\vec{\mathbf{F}}}{\mathbf{m}} =$ Acceleration under central force directed towards or away from the fixed point.




**b)** For central force $\vec{\mathbf{r}} \times \vec{\mathbf{F}} = \mathbf{0}$ , and then we have $\vec{\mathbf{r}} \times \vec{\mathbf{f}}(\mathbf{r}) = \mathbf{0}$ or $\vec{\mathbf{r}} \times \frac{\mathbf{d^2\vec{r}}}{\mathbf{dt^2}} = \mathbf{0}$

Again

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{r}} \times \frac{\mathbf{d\vec{r}}}{\mathbf{dt}}\right) = \left(\frac{\mathbf{d\vec{r}}}{\mathbf{dt}} \times \frac{\mathbf{d\vec{r}}}{\mathbf{dt}}\right) + \left(\vec{\mathbf{r}} \times \frac{\mathbf{d^2\vec{r}}}{\mathbf{dt^2}}\right)$$
$$= \mathbf{0} + \vec{\mathbf{r}} \times \frac{\mathbf{d^2\vec{r}}}{\mathbf{dt^2}}.$$

So we have $\frac{\mathbf{d}}{\mathbf{dt}}\left(\vec{\mathbf{r}} \times \frac{\mathbf{d\vec{r}}}{\mathbf{dt}}\right) = \mathbf{0}$

Integrating, we have, $\left(\vec{\mathbf{r}} \times \frac{\mathbf{d\vec{r}}}{\mathbf{dt}}\right) = \mathbf{h}$, where $\mathbf{h}$ is a constant. Again we have $\frac{1}{2}\left(\vec{\mathbf{r}} \times \frac{\mathbf{d\vec{r}}}{\mathbf{dt}}\right) = \frac{1}{2}\mathbf{h}$

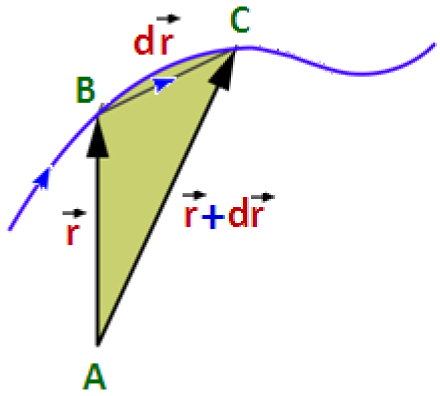
So we can conclude that for particle motion under central force, the area swept out by the radius vector in unit time for such particle motion will be constant. More clearly, the areal velocity is constant for motion of particle under a central force.

c) Since $\vec{r} \times \vec{F} = 0$ (i.e. the moment of this force about the fixed point is zero), the torque effective on that particle moving under central force will be zero.

That is $\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \implies \vec{L} = \text{Constant}$. This gives that the angular momentum of particle motion under central force will remain conserved. Again we have

$$\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = h \implies m\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = mh \implies \left(\vec{r} \times m\frac{d\vec{r}}{dt}\right) = mh \implies \vec{L} = \text{Constant}$$

d) It should also be noted that the vector product $\left(\vec{r} \times \frac{d\vec{r}}{dt}\right)$ is a vector normal to the plane determined by the fixed directions of $\vec{r}$ and $d\vec{r}$. The vector $m\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \left(\vec{r} \times m\frac{d\vec{r}}{dt}\right)$ is called the moment of momentum or the angular momentum about the fixed point. Thus we may conclude that for motion of a particle under a central force field, the angular momentum vector remains constant, and the orbit of the particle is such that for any value of $\vec{r}$ and $\frac{d\vec{r}}{dt}$ on it, there is a constant vector associated, and hence the orbit described will lie in a plane. Hence the motion of a particle under central force field must occur always in a plane.




e) For such central force $\vec{F} = |\vec{F}|(\pm\hat{r}) = \pm\frac{F}{r}\vec{r}$ we have $\vec{\nabla} \times \vec{F} = \pm\vec{\nabla} \times \left(\frac{F}{r}\vec{r}\right)$. Hence we get $\vec{\nabla} \times \vec{F} = \pm\left[\vec{\nabla}\left(\frac{F}{r}\right) \times \vec{r} + \left(\frac{F}{r}\right)\vec{\nabla} \times \vec{r}\right] = \pm\left[\vec{\nabla}\left(\frac{F}{r}\right) \times \vec{r} + 0\right] = \pm\left[\left\{\frac{1}{r}\vec{\nabla}(F) + F\vec{\nabla}\left(\frac{1}{r}\right)\right\} \times \vec{r}\right]$

Finally we get $\vec{\nabla} \times \vec{F} = \pm\left[\left\{\frac{1}{r}.\frac{F'(r)}{r}\vec{r} - F\left(\frac{\vec{r}}{r^3}\right)\right\} \times \vec{r}\right] = 0$

Thus central force is always 'irrotational' and hence 'conservative' and in that case we must have $\vec{F} = -\vec{\nabla}(U)$ and also usually $F = -\frac{\partial U}{\partial r}$ where $U$ is the potential energy.

Thus for $\vec{F}(r)$ to be the central force, since it is always effective in radial sense, the equation of motion of a body of mass $m$ under central force is given by $ma_r = m(\ddot{r} - r\dot{\theta}^2) = F(r)$ where $a_r = (\ddot{r} - r\dot{\theta}^2)$ denotes the radial acceleration.

But also since there is no force component in transverse sense for particle motion along any curve (also called 'orbit') under central force, we have $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ . Thus we get $r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \implies \frac{d}{dt}(r^2\dot{\theta}) = 0 \implies r^2\dot{\theta} = \text{Constant} = h$ (say)

So we get $m\dot{\theta} = \frac{mh}{r^2} = \frac{H}{r^2}(H = mh = \text{another Constant})$ . Hence we get from the force equation as given above

$$m\left(\ddot{r} - \frac{H^2}{m^2r^3}\right) = F(r) \quad \text{or} \quad m\ddot{r} = \frac{H^2}{mr^3} + F(r). \text{-----------} (1)$$

Since a central force is a conservative field it can be derived from a potential energy U which is a function of r alone i.e. $F(r) = -\frac{\partial U}{\partial r}$ . Se we get from equation (1)

$$m\ddot{r} = \frac{H^2}{mr^3} - \frac{\partial U}{\partial r} = -\frac{1}{2}\left(\frac{\partial}{\partial r}\right)\left(\frac{H^2}{mr^2}\right) - \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\left[\frac{1}{2}\left(\frac{H^2}{mr^2}\right) + U\right].$$

Multiplying both sides of the equation by $\dot{r}$, we have

$$m\dot{r}\ddot{r} = -\dot{r}\left(\frac{\partial}{\partial r}\right)\left[\frac{1}{2}\left(\frac{H^2}{mr^2}\right) + U\right] \quad \text{or} \quad \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left[\frac{1}{2}\left(\frac{H^2}{mr^2}\right) + U\right]$$

$$\text{or} \quad \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(\frac{H^2}{mr^2}\right) + U = \text{Constant}. \text{------------} (2)$$

Now the Kinetic energy of that moving particle under central force is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{H^2}{r^4m^2} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(\frac{H^2}{mr^2}\right).$$

So we have from equation (2)

Total energy $E = \text{Kinetic energy + potential energy} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(\frac{H^2}{mr^2}\right) + U = \text{Constant}.$

This proves the conservation of energy for a central force field.